1. **Diagonalizable Operators on Invariant Subspaces.** Let $T : V \to V$ be a diagonalizable operator on a finite dimensional vector space $V$ over a field $\mathbb{F}$. Suppose $W \subset V$ is a $T$-invariant subspace. Show that $T|_W$ is diagonalizable by considering the minimal polynomial $m_{T|_W}$.

   Hint: Use the fact that $T$ is diagonalizable if and only if its minimal polynomial is the product of distinct monic linear polynomials.

2. **Simultaneous Diagonalizability.** Let $S, T : V \to V$ be linear transformations. We say that $S, T$ are simultaneously diagonalizable if there exists a direct sum decomposition $V = \bigoplus_{i=1}^k V_i$, and scalars $\lambda_i, \mu_i \in \mathbb{F}$, such that $T|_{V_i} = \lambda_i \text{Id}_{V_i}$, $S|_{V_i} = \mu_i \text{Id}_{V_i}$ for $i = 1, \ldots, k$.

   (a) Assume that $S, T$ are diagonalizable. Show that $ST = TS$ if and only if $S, T$ are simultaneously diagonalizable. Recall, that you showed one direction of this in a previous HW.

   (b) Let $V$ be a finite dimensional vector space over a field $\mathbb{F}$. Denote by $L(V)$ the set of linear transformations from $V$ to itself.

      (i) Show that $L(V)$ is an algebra over $\mathbb{F}$ in a natural way. That is, it has a natural addition and multiplication.

      (ii) Let $C \subset L(V)$ be a subalgebra, i.e. closed under multiplication and addition, consisting of diagonalizable operators. Show that all elements of $C$ are simultaneously diagonalizable if and only if $C$ is commutative.

   Here simultaneously diagonalizable means there exists a decomposition $V = \bigoplus_{i=1}^k V_i$ such that $T|_{V_i} = \lambda_T \cdot \text{Id}_{V_i}$ for any $T \in C$. Recall, an algebra $C$ is commutative if for any $C_1, C_2 \in C$ we have $C_1 C_2 = C_2 C_1$. 


3. **Nilpotent Operators.** Let $T : V \to V$ be a linear transformation on a finite dimensional vector space $V$ over $\mathbb{F}$. We say that $T$ is **nilpotent** if there exists a flag of subspaces of $V$,

$$\{0_V\} = V_0 \subset V_1 \subset \cdots \subset V_k = V,$$

such that $T(V_i) \subset V_{i-1}$ for all $i = 1, \ldots, k$. We define the nilpotency degree of a nilpotent operator $T$ as the smallest positive integer $m$ such that $T^m = 0$.

(a) Show the following operators are nilpotent by constructing a flag as above. In each case, compute the nilpotency degree of the given operator.

(i) The derivative operator $D$ on $F_{\leq 3}[x]$, the space of polynomials over $F$ of degree at most 3.

(ii) The operator $T_A : \mathbb{R}^5 \to \mathbb{R}^5$, defined by multiplication by

$$A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

(b) Show $T$ is nilpotent if and only if $T^k = 0$ for some integer $k > 0$.

(c) Show $T$ is nilpotent if and only if there is a basis $B$ of $V$ such that $[T]_B$ is a strictly upper-triangular matrix, i.e. all entries on and below the diagonal are 0.

(d) Let $V$ be a vector space over $\mathbb{F}$. Consider a flag of the form:

$$\mathcal{F} : \{0_V\} = V_0 \subset V_1 \subset \cdots \subset V_k = V.$$

Denote by $N_{\mathcal{F}}$ the set of all linear transformations $T : V \to V$, such that $T(V_i) \subset V_{i-1}$ for $i = 1, \ldots, k$. Show $N_{\mathcal{F}}$ is a subalgebra of $L(V)$, i.e. it is closed under addition and composition.

(e) Let $\mathcal{N} \subset L(V)$ be a maximal collection of commuting nilpotent operators. Show that $\mathcal{N}$ is a subalgebra.

4. **Generalized Eigenspaces.** Let $T : V \to V$ be a linear transformation on a finite dimensional vector space $V$ over $\mathbb{F}$. Suppose that $m_T(x) = (x - \lambda_1)^{r_1} \cdots (x - \lambda_s)^{r_s}$. Denote by $W_{\lambda_k}$ the generalized eigenspace of $V$ associated to $\lambda_k$, that is, $W_{\lambda_k}$ is the set of $v \in V$ such that $(T - \lambda_k \cdot Id_V)^m v = 0$ for some $m > 0$. Show that

$$W_{\lambda_k} = \ker(T - \lambda_k \cdot Id_V)^{r_k}.$$

5. **Similar Transformations.** Let $T, S : V \to V$ be two transformations of a finite dimensional vector space $V$ over $\mathbb{C}$.

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(a) Suppose \( \text{dim}(V) = 2 \) and that \( m_T = m_S \). Is it true that \( T \) and \( S \) are similar?
(b) Suppose \( \text{dim}(V) = 3 \) and that \( m_T = m_S \). Is it true that \( T \) and \( S \) are similar?
(c) Suppose \( \text{dim}(V) = 3 \) and that both \( m_T = m_S \) and \( p_T = p_S \). Is it true that \( T \) and \( S \) are similar?

**Remark**
The grader and the Lecturer will be happy to help you with the homework. Please visit office hours.

**Good luck!**