1. Exercises

(1) Let $S$ be a subring of $R$.
   (a) Give an example where $R$ is noetherian and $S$ is not noetherian.
   (b) Assume there is an $S$-module map $\phi: R \to S$ which is the identity when restricted to $S$. Show that $S$ is noetherian if $R$ is noetherian.

(2) Let $R$ be a ring and let $\phi: R^n \to R^n$ be a linear map where $n$ is finite.
   (a) Show that $\det \phi \in \text{ann}(\text{coker} \phi)$.
   (b) Give an example where $\det \phi$ is a nonzerodivisor but $\text{ann}(\text{coker} \phi)$ is not equal to the ideal generated by $\det \phi$.

(3) A ring is reduced if its nilradical is 0.
   (a) Show that a ring $R$ is reduced if and only if $R_m$ is reduced for all maximal ideals $m$.
   (b) Give an example of a field $k$ and a reduced $k$-algebra $R$ such that $R \otimes_k k$ is not reduced where $k$ is an algebraic closure of $k$.

(4) Let $k$ be a field. In each case, verify that $R$ is an integral domain and describe, as explicitly as possible, its normalization.
   (a) $R = k[x,y]/(x^4 - y^3)$
   (b) $R = k[x,y,z]/(x^2 - yz)$

(5) Let $R$ be a noetherian ring and let $M$ be a finitely generated $R$-module. Recall in HW3 #4, we defined the dual $M^\vee = \text{Hom}_R(M, R)$ and the map $\sigma_M: M \to (M^\vee)^\vee$ by $
   \sigma_M(m)(f) = f(m)$ (where $m \in M$ and $f \in M^\vee$).
   (a) Show that $\sigma_M$ is injective if and only if $M$ is isomorphic to a submodule of $R^\oplus n$ for some finite $n$.
   (b) If $\sigma_M$ is bijective, show that there exists a homomorphism $\phi: R^\oplus n \to R^\oplus m$ for some finite $n, m$ such that $M \cong \ker \phi$.

(6) Let $S \subset R$ be an integral extension.
   (a) Show that $\dim S = \dim R$.
   (b) Assume that $R$ is a finitely generated $S$-module. Let $p \subset S$ be a prime ideal. Show that there are only finitely many prime ideals $q \subset R$ such that $q \cap S = p$. 