Given vector spaces $V \neq 0$, $W \neq 0$

Given bases $X \subset V$, $Y \subset W$

Ex 7 yields

Coalgebra $V \otimes W$

Prop 9 yields

Coalgebra $V \otimes W$

Using the basis $\{ x \otimes y \mid x \in X, y \in Y \}$ in $V \otimes W$

Ex 7 yields

Coalgebra $V \otimes W$

Compare II, III
LEM 10 the above configurations II, III are the same.

pf (ex)

detail)

I

\[ \Delta_v : V \rightarrow \nu \varepsilon V \]
\[ x \rightarrow x \nu x \]

\[ \varepsilon_v : V \rightarrow 1 \]
\[ x \rightarrow 1 \]

\[ \Delta_w : W \rightarrow w \nu w \]
\[ y \rightarrow y \nu y \]

\[ \varepsilon_w : W \rightarrow 1 \]
\[ y \rightarrow 1 \]

II

\[ \Delta : \nu \varepsilon w \rightarrow \nu \varepsilon \varepsilon w \varepsilon w \]
\[ x \nu y \rightarrow x \nu \varepsilon \varepsilon y \varepsilon y \rightarrow (x \varepsilon y) \varepsilon (y \varepsilon y) \]

\[ \varepsilon : \nu \varepsilon w \rightarrow k \varepsilon k \rightarrow 1 \]
\[ x \varepsilon y \rightarrow 1 \varepsilon 1 \rightarrow 1 \]

III

\[ \Delta : \nu \varepsilon w \rightarrow \nu \varepsilon \varepsilon w \varepsilon w \]
\[ x \nu y \rightarrow (x \varepsilon y) \varepsilon (x \varepsilon y) \]

\[ \varepsilon : \nu \varepsilon w \rightarrow k \]
\[ x \varepsilon y \rightarrow 1 \]
Given fin dim' algebras $A, B$.

Get coalgebras $A^*, B^*$.

Given algebra morphism $\varphi: A \to B$.

LEM II The transpose

$\varphi^*: B^* \to A^*$

is a coalgebra morphism.

pf

$\forall f \in A^*$ write

$\Delta_{A^*}(f) = \sum_i f'_i \otimes f''_i$

$\forall g \in B^*$ write

$\Delta_{B^*}(g) = \sum_j g'_j \otimes g''_j$

$\forall a_1, a_2 \in A$

$f(a_1, a_2) = \sum_i f'_i(a_1) f''_i(a_2)$

$\forall b_1, b_2 \in B$

$g(b_1, b_2) = \sum_j g'_j(b_1) g''_j(b_2)$

Take $b_1 = \varphi(a_1), \quad b_2 = \varphi(a_2)$

Also $\forall a \in A$

$\langle g, \varphi(a) \rangle = \langle \varphi^*(g), a \rangle$
Check this diagram commutes:

\[ \begin{array}{ccc}
B^* & \rightarrow & A^* \\
\Delta_{B^*} & \downarrow & \Delta_{A^*} \\
B^* \otimes B^* & \rightarrow & A^* \otimes A^* \\
\psi^* \otimes \psi^* & & \\
\end{array} \]

\[ g \rightarrow \psi^*(g_1) \]

\[ \Delta_{A^*}(\psi^*(g_1)) \]

\[ \sum_{\tilde{g}_0} \tilde{g}_0 \otimes g''_0 \rightarrow \sum_{\tilde{g}_0} \psi^*(g_0) \otimes \psi^*(g''_0) \]

Requiring:

\[ \psi^*(g)(a, a_2) = \sum_{\tilde{g}_0} \left( \frac{\psi^*(\tilde{g}_0)}{\tilde{g}_0} \otimes \frac{\psi(\tilde{g''}_0)}{\tilde{g''}_0} \right) \]

\[ \ll \]

\[ \langle \psi^*(g), a, a_2 \rangle \]

\[ \ll \]

\[ \langle g, \psi(a, a_2) \rangle \]

\[ \ll \]

\[ \langle g, \psi(a_1) \psi(a_2) \rangle \]

\[ \ll \]

\[ \langle g, g(b, b_2) \langle \psi(a_1), \psi(a_2) \rangle \rangle \]

\[ \ll \]

\[ \sum_{\tilde{g}_0, \tilde{g}_2} \tilde{g}_0 \tilde{g}_2 (b, b_2) \]
check this diagram commutes:

\[ 
\begin{array}{ccc}
\beta^* & \rightarrow & A^* \\
\downarrow & & \downarrow \varepsilon_{A^*} \\
K & \rightarrow & k \\
\end{array}
\]

\[ 
\begin{array}{ccc}
g & \rightarrow & \psi^*(g) \\
\downarrow & & \downarrow \\
g(1_\beta) & \rightarrow & \psi^*(g)(1_\alpha) \\
\end{array}
\]

\[ 
\begin{array}{ccc}
g(1_\alpha) & = & \psi^*(g)(1_\alpha) \\
\langle g, 1_\beta \rangle & = & \langle \psi^*(g), 1_\alpha \rangle \\
\end{array}
\]
Given coalgebras $A$ and $B$.

Get algebras $A^* \otimes B^*$.

Given coalgebra morphism

\[ \psi : A \rightarrow B \]

**LEM 12** the transpose

\[ \psi^* : B^* \rightarrow A^* \]

is an algebra morphism.

**Proof**

Let $a \in A$. Write

\[ \Delta_A (a) = \sum_i a'_i \otimes a''_i \]

Write

\[ b = \psi(a) \]

\[ b'_i = \psi(a'_i) \]

\[ b''_i = \psi(a''_i) \]

We have

\[
\begin{array}{ccc}
A & \xrightarrow{\psi} & B \\
\downarrow \Delta_A & & \downarrow \Delta_B \\
A \otimes A & \rightarrow & B \otimes B
\end{array}
\]

Given

\[ \Delta_B (b) = \sum_i b'_i \otimes b''_i \]
By construction, \( \forall f, g \in A^* \)

\[
(fg)(a) = \sum_i f(a_i') g(a_i'')
\]

Also \( \forall F, G \in B^* \)

\[
(FG)(b) = \sum_i F(b_i') G(b_i'')
\]

Also \( \forall H \in B^* \)

\[
\langle H, \psi(a) \rangle = \langle \psi(H), a \rangle
\]

Check \( \psi^*(FG) = \psi^*(F) \psi^*(G) \)

Require \( \psi^*(FG)(a) = (\psi^*(F) \psi^*(G))(a) \)

\[
\langle \psi^*(FG), a \rangle \leq \sum_i \psi^*(F)(a_i') \psi^*(G)(a_i'')
\]

\[
\langle FG, \psi(a) \rangle \leq \sum_i F(b_i') G(b_i'')
\]
Check
\[ \varphi^* (1_{B^*}) = 1_{A^*} \]

Recall
\[ 1_{B^*} = \varepsilon_B \quad 1_{A^*} = \varepsilon_A \]

We have
\[ \begin{array}{c}
\varphi \\
A \rightarrow B \\
\varepsilon_A \downarrow \downarrow \varepsilon_B \\
k \rightarrow k \\
id
\end{array} \]

\[ \varepsilon_A (a) = \varepsilon_B (\varphi(a)) \]

\[ \varphi^* (\varepsilon_B) = \varepsilon_A \]

\[ \varphi^* (\varepsilon_B) (a) = \varepsilon_A (a) \]

\[ \left< \varphi^* (\varepsilon_B), a \right> \]

\[ \left< \varepsilon_B, \varphi (a) \right> \]

\[ \varepsilon_B (\varphi (a)) \]

\[ \varepsilon_A (a) \]

\[ \square \]
Given coalgebras $A, B$

Given coalgebra morphism

$\varphi : A \to B$

Describe $\text{Image}(\varphi), \text{ker}(\varphi)$

$\varphi$

Consider $C$

$\Delta_B(C) \leq C \otimes C$

$C$ becomes coalgebra with

$\Delta_C = \Delta_B / C$

$\varepsilon_C = \varepsilon_B / C$

Also

$\psi : A \to C$

is coalgebra morphism.
Consider $I$

\[ I \xrightarrow{\psi} 0 \]

\[ \Delta_a \downarrow \Delta_b \]

\[ \psi \circ \psi \]

\[ \Delta_a(I) \subseteq \ker(\psi \circ \psi) = I \otimes A + A \otimes I \]

\[ I \xrightarrow{\psi} 0 \]

\[ \varepsilon_a \downarrow \varepsilon_b \]

\[ 0 \xrightarrow{id} 0 \]

\[ \varepsilon_a(I) = 0. \]

**DEF 13** Given coalgebra $A$

An **ideal** of $A$ is a subspace $I$ of $A$

\[ \Delta(I) \subseteq I \otimes A + A \otimes I, \]

\[ \varepsilon(I) = 0. \]
Given coalgebra \( A \)

Given coideal \( I \) of \( A \)

Consider quotient vector space \( \tilde{A} = A/I \)

We now turn \( \tilde{A} \) into a coalgebra.

Consider linear map

\[
\mathrm{can}: \quad A \rightarrow \tilde{A} \\
\quad a \mapsto a + I
\]

\( \exists \) unique linear map

\[
\tilde{\Delta} : \quad \tilde{A} \rightarrow \tilde{A} \otimes \tilde{A}
\]

\( \Delta \) makes this diagram commute:

\[
\begin{array}{c}
\Delta \\
\downarrow \\
A \otimes A
\end{array}
\xrightarrow{\mathrm{can}}
\begin{array}{c}
\tilde{\Delta} \\
\downarrow \\
\tilde{A} \otimes \tilde{A}
\end{array}
\]

Indeed \( \forall a \in A \)

\[
\tilde{\Delta}(a) = \sum_i (a_i + I) \otimes (a_{i}^{''} + I)
\]

where \( \Delta(a) = \sum_i a_i \otimes a_{i}^{''} \)
Also, there is a unique linear map

$$\tilde{\varepsilon}: \tilde{A} \to K$$

s.t.

$$\varepsilon \downarrow \quad \tilde{\varepsilon} \downarrow \tilde{\varepsilon}$$

commutes

The maps $\tilde{\Delta}$, $\tilde{\varepsilon}$ turn $\tilde{A}$ into a coalgebra.

By Constr. $\text{can}: A \to \tilde{A}$ is coalgebra morphism.
Sweedler notation

Given coalgebra $C$

For $c \in C$ write

$$\Delta(c) = \sum_i c_i'^{\prime} c_i''$$

Coassociativity:

$$c = \sum_i \varepsilon(c_i'^{\prime}) c_i'' = \sum_i c_i'^{\prime} \varepsilon(c_i'')$$

In, suppress index $i$ and write

$$\Delta(c) = \sum_{(c)} c'^{\prime} c''$$

** becomes

$$c = \sum_{(c)} \varepsilon(c'^{\prime}) c'' = \sum_{(c)} c'^{\prime} \varepsilon(c'')$$

Coassociative property becomes

$$\sum_{(c)} \Delta(c') \otimes c'' = \sum_{(c)} c'^{\prime} \otimes \Delta(c'')$$

We abbreviate this common value by

$$\sum_{(c)} c'^{\prime} \otimes c''$$

(Long form)

or

$$\Delta^{[2]}(c)$$

(Short form)
Similarly,

\[ \sum \Delta(c') \otimes c'' \otimes c''' = \sum c' \otimes d(c'') \otimes c''' = \sum c' \otimes c'' \otimes d(c''') \]

is abbreviated by

\[ \sum c' \otimes c'' \otimes c''' \otimes civ \]  

(Long form)

or

\[ \Delta^{(iv)}(c) \]  

(Short form)

etc.

Consider algebra \( C^k \).

For \( f, g \in C^k \) and \( c \in C \),

\[ (f \otimes g)(c) = \sum f(c') \otimes g(c'') \]

Given algebra \( A \), consider coalgebra \( A^k \).

For \( f \in A^k \) and \( a, b \in A \),

\[ f(ab) = \sum f'(a) \otimes f''(b) \]