Recall $M_2$ has basis

$$a_{i,b}^t, \quad i,j \in \mathbb{N}$$

Recall the sum

$$M_2[t_i, k_j] = M_2 + M_2t + M_2t^2 + \cdots$$

is direct. Therefore $M_2[t_i, k_j]$ has basis

$$a_{i,b}^t k, \quad i,j,k \in \mathbb{N}$$

By contr $M_3$ is spanned by

$$a_{i,b}^t k, \quad i,j,k \in \mathbb{N}$$

$\theta$ sends

$$a_{i,b}^t k \rightarrow a_{i,b}^t k, \quad i,j,k \in \mathbb{N}$$

Therefore $\theta$ is a basis for $M_3$ and $\theta$ is an iso.

We have shown $M_3 \cong M_2[t_i, k_j]$ in an Ore ext.

$M_2$
Show $M_4$ is an Ore ext of $M_3$

\[ \exists \text{ alg morphism } \alpha : M_3 \to M_4 \text{ that sends } \]
\[ a \to a, \quad b \to qb, \quad c \to qc \]

(since $a = a$, $b = qb$, $c = qc$ satisfy the defining relations for $M_3$)

Obs $\alpha$ is a bijection.

Show $\exists$ $\lambda$-derivation $\delta : M_3 \to M_3$ that sends
\[ a \to (q^{-i} b c), \quad b \to 0, \quad c \to 0 \]

$\exists$ lin map $\delta : M_3 \to M_3$ that sends
\[ a^i b^j c^k \to q^{-i} (q^{-i} q^{-i}) a^i \cdot b^j \cdot c^k \quad \text{ker} \]

for $i, j, k \in \mathbb{N}$. 

Show $\delta$ is $\lambda$-der.

Show
\[ \delta(xy) = \delta(x) y + \alpha(x) \delta(y) \quad \forall x, y \in M_3 \]
\[ x = a_i b^k \quad y = a_r b^e c^t \]

**Obs**
\[ x = a_i b^{r + e} c^{k + t} \]

\[ \delta(x) = q^{r + e + t + i} \left( q^{a_i - q^r} \right) a^{r + e} b^{r + e} c^{k + t} \]

\[ \delta(x) = q^{r + e} \left( q^{a_i - q^r} \right) a^{r + e} b^{r + e} c^{k + t} \]

\[ \delta(x) = q^{r + e + t + i} \left( q^{a_i - q^r} \right) a^{r + e} b^{r + e} c^{k + t} \]

\[ \delta(x) = q^{a_i} b^k \]

\[ \delta(y) = q^{r + e} \left( q^{a_i - q^r} \right) a^{r + e} b^{r + e} c^{k + t} \]

\[ \delta(x) \delta(y) = q^{r + e + t + i} \left( q^{a_i - q^r} \right) a^{r + e} b^{r + e} c^{k + t} \]

By these comments
\[ \delta(xy) = \delta(x) \delta(y) + \delta(x) \delta(y) \]

So \( \delta \) is an under \( q \) M3
For the Ore ext $M_3[t, x, s]$

\[
\begin{align*}
t_a &= s(a) + a(s) = (g - q^2)bc + qa \\
t_b &= s(b) + a(b) = qtb \\
t_c &= s(c) + a(c) = qtc
\end{align*}
\]

So $\exists$ alg morph

\[
\Psi : M_4 \rightarrow M_3[t, x, s]
\]

But sends

\[
a \mapsto a, \quad b \mapsto b, \quad c \mapsto c, \quad d \mapsto t
\]

Show $\Psi$ is ISO

Recall $M_3$ has a basis

\[
a, b, c, k \quad i, j, k \in \mathbb{N}
\]

Recall the sum

\[
M_3[t, x, s] = M_3 + M_3t + M_3t^2 + \ldots
\]

is direct
So $M_3[6i,6j]$ has a basis

$$\begin{array}{c}
a^i b^j c^k d^l \\
i, j, k, l \in \mathbb{N}
\end{array}$$

Using the rule $\ast$, we find $M_Y$ is spanned by

$$\begin{array}{c}
a^i b^j c^k d^l \\
i, j, k, l \in \mathbb{N}
\end{array}$$

$\Phi$ sends

$$\begin{array}{c}
a^i b^j c^k d^l \to a^{i+k} b^{j+l} \\
i, j, k, l \in \mathbb{N}
\end{array}$$

Hence $\ast \ast$ is a basis for $M_4$ and $\Phi$ is an isomorphism.

We have shown that $M_Y = M_3[6i,6j]$ is an

One cell of $M_3$.

---

**COR 8**

(i) $M_9[12]$ has a basis

$$\begin{array}{c}
a^i b^j c^k d^l \\
i, j, k, l \in \mathbb{N}
\end{array}$$

(ii) $M_{12}$ has no non-trivial $\text{O}$-divisors

(iii) $M_{12}$ is Noetherian
LEM 9 \quad \exists \quad \text{algebra iso} \quad M_q(2)^{\text{op}} \to M_q^{-1}(2)

that sends
\begin{align*}
a &\to a, \\
b &\to b, \\
c &\to c, \\
d &\to d
\end{align*}

pf. Compare the defining relations for $M_q(2)^{\text{op}}$, $M_q^{-1}(2)$

$M_q(2)^{\text{op}}$:
\begin{align*}
ab &= qba \\
ac &= qca \\
cb &= cb
\end{align*}
\begin{align*}
cd &= qdc \\
bd &= qdb \\
da-ad &= (q^{-1}-q)cb
\end{align*}

$M_q^{-1}(2)$:
\begin{align*}
b a &= q^{-1}ab \\
ca &= q^{-1}ac \\
bc &= cb
\end{align*}
\begin{align*}
dc &= q^{-1}cd \\
db &= q^{-1}bd \\
ad-ad &= (q^{-1}-q)bc
\end{align*}
the \textbf{q-det}

Moebius: \quad \text{In } M(2) \text{ recall}

\[ s = x_{11} x_{22} - x_{12} x_{21} = ad - bc \]

satisfies \quad \Delta(s) = s \circ s

For \ M_\mathbb{K}(2), \text{ pick } \psi \in \mathbb{K} \text{ and consider}

\[ s = \psi d - \psi bc \quad \in \ M_\mathbb{K}(2) \]

Find \( \psi \) s.t.

\[ \Delta(s) = s \circ s \]

Recall \( \Delta \)

\[ \Delta : \quad M_\mathbb{K}(2) \rightarrow M_\mathbb{K}(2) \otimes M_\mathbb{K}(2) \]

\[ a \rightarrow a \otimes a + b \otimes c \quad = A \]

\[ b \rightarrow a \otimes b + b \otimes b \quad = B \]

\[ c \rightarrow c \otimes a + d \otimes c \quad = C \]

\[ d \rightarrow c \otimes b + d \otimes d \quad = 0 \]
\[ \Delta(x) = AD - x \cdot BC \]

\[
\begin{array}{c|c|c}
& \text{coef} & \text{term} \\
\hline
ac & ab - 2ba & = ab(1 - 2x) \\
ad & ad - xbc & = s \\
bc & cb - xda & = -2s + 6c(1-x)(1+x^2) \\
bd & cd - xdc & = cd(1 - x^2) \\
\end{array}
\]

\[ \text{[take } x = 1 - 2x]\]

\[ = s \otimes s \quad \text{provided } x = 9\]

Obs.
\[ ad - q^2 bc = da - qbc \]

Call this common value \( \text{det}_q \)

We have
\[ \Delta(\text{det}_q) = \text{det}_q \otimes \text{det}_q \]

Obs.
\[ \epsilon(\text{det}_q) = 1 \]
LEM 10 The element $d_{12}$ is in the center of $M_4(2)$.

pf Using the defining relations for $M_4(2)$, one checks that $ad - a + bc$ commutes with each of $a, b, c, d$. □
Given an algebra $R$, an $R$-point of $M_2(2)$ is a matrix

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
$$

$A, B, C, D \in R$

subject to

$$
OA = qAB, \\
CA = qAC, \\
BC = CB,
$$

$$
DC = qCD, \\
DB = qBD, \\
AO - OA = (q^{-q}) BC
$$

So if $A, B, C, D \in R$

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
$$

is an $R$-point of $M_2(2)$

if

$\exists$ an $R$-algebra morphism $M_2(2) \to R$ that sends

$a \to A, \quad b \to B, \quad c \to C, \quad d \to D$

Given an $R$-point of $M_2(2)$:

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
$$

define

$$
\text{Det}_q \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = AD - q^2 BC
$$
LEM II  Given algebra $R$

Given $R$-points $X, Y$ of $M_2(R)$:

$$X = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \quad Y = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

Assume each entry of $X$ commutes with each entry of $Y$.

Then

(i) Matrix product $XY$ is an $R$-pt of $M_2(R)$

(ii) $\text{det}_R(XY) = \text{det}_R(X) \cdot \text{det}_R(Y)$

pf We have

$$XY = \begin{pmatrix} AA' + BC' & AB' + BD' \\ CA' + DC' & CB' + DD' \end{pmatrix}$$

Recall

$$\Delta : M_2(R) \rightarrow M_2(R) \otimes M_2(R)$$

is a $\text{alg}$ morphism.

Also, by assumption $\exists$ a $\text{alg}$ morphism

$$M_2(R) \otimes M_2(R) \rightarrow R$$
Consider any morphism

\[ \varphi : M_4(2) \rightarrow M_4(2) \otimes M_4(2) \rightarrow R \]

\[ \Delta \rightarrow \ast \]

(i) \( \varphi \) sends

\[ a \rightarrow a \otimes a \rightarrow A \otimes A' = \left( \begin{array}{cc} 0_{1 \times 1} & 0_{1 \times 1} \\ 0_{1 \times 1} & 0_{1 \times 1} \end{array} \right) \]

\[ b \rightarrow b \otimes b \rightarrow B \otimes B' = \left( \begin{array}{cc} 0_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 2} & 0_{2 \times 2} \end{array} \right) \]

\[ c \rightarrow c \otimes c \rightarrow C \otimes C' = \left( \begin{array}{cc} 0_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 2} & 0_{2 \times 2} \end{array} \right) \]

\[ d \rightarrow d \otimes d \rightarrow D \otimes D' = \left( \begin{array}{cc} 0_{2 \times 2} & 0_{2 \times 2} \\ 0_{2 \times 2} & 0_{2 \times 2} \end{array} \right) \]

So \( XY \) is an \( R \)-point of \( M_4(2) \)

(ii) \( \varphi \) is any morphism so

\[ \Theta(\delta x \gamma) = \Theta(a \otimes a - b \otimes b) = a(a) b(b) - b(a) a(b) \]

\[ = \delta \delta \delta \gamma \gamma \]

\[ \text{But } \varphi \text{ sends} \]

\[ \delta x \gamma \Delta \rightarrow \delta x \gamma \otimes \delta x \gamma \rightarrow \delta \delta \delta \gamma \gamma \]

Result follows.
Given algebra $R$

We have some comments about $R$-points.

- $(10)$ is an $R$-point in $M_2(2)$

Given $R$-points in $M_2(2)$:

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
$$

- For $a, b, r, s \in K$ s.t. $rs = s^r$,
  $$(aA, bB)$$
  is an $R$-pt in $M_2(2)$

- The transpose
  $$
  \begin{pmatrix}
  A & C \\
  B & D
  \end{pmatrix}
  $$
  is an $R$-pt in $M_2(2)$

- The matrix
  $$
  \begin{pmatrix}
  B & C \\
  A & D
  \end{pmatrix}
  $$
  is an $R$-point in $M_{4\times 4}(2)$

- The matrix
  $$
  \begin{pmatrix}
  0 & B \\
  C & A
  \end{pmatrix}
  $$
  is an $R^{op}$-point in $M_2(2)$
The matrix
\[
\begin{pmatrix}
D & -qB \\
-q^2C & A
\end{pmatrix}
\]
is an $R$-pt in $M_q(2)$ and an $R^{op}$-pt in $M_q(2)$.

0

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
D & -qB \\
-q^2C & A
\end{pmatrix}
= \begin{pmatrix}
A_0 - q^2BC & 0 \\
0 & AD - q^2BC
\end{pmatrix}
\]

= \begin{pmatrix}
D & -qB \\
-q^2C & A
\end{pmatrix}
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}