VII. 3 Action of $U_q(sl(2))$ on the Quantum Plane

This section is the quantum version of VII.6. We start with a few generalities on skew-exponential skew-derivations of an algebra $A$.

Let $A$ be an algebra. For $a \in A$, denote its left and right multiplication by the elements $a$ by $a_l$ and $a_r$, respectively.

If $\sigma$ is an automorphism of $A$, the algebra $A$, then it's easy to verify:

\[ \sigma(a) = \sigma(a_l) \sigma(a_r) \]

Check: $a \cdot a : (\sigma(a) \cdot x) = \sigma(a) \cdot (x) = \sigma(a_l) \cdot (x)$

Similarly, we have $\sigma(a) \cdot a = \sigma(a_l) \cdot a_r$.

Given two automorphisms $\sigma$ and $\tau$ of the algebra $A$,
a linear endomorphism $S$ of $A$ is called a $(\sigma, \tau)$-derivation if $S(a \cdot a') = \sigma(a) S(a') + S(a) \tau(a')$ for all $a, a' \in A$.

Note that this relation is equivalent to $S(a) = \sigma(a) S(a) + S(a) \tau(a)$.

Check: $S(a \cdot a') = S(a \cdot a') = \sigma(a) S(a') + S(a) \tau(a')$ for all $a, a' \in A$.

Similarly, it's equivalent to $S(a) = \sigma(a) S(a) + S(a) \tau(a')$ for all $a, a' \in A$. 

\[ \sigma(a) = \sigma(a_l) \sigma(a_r) \]
Remark: It's well-known that, if $s$ is a derivation of a commutative algebra, then $as$ is a derivation too.

Check: $(as)(ab') = a(s(ab'))$

- $a(0)(s(a'b') + a(s(a')b'))$
- $a(s(a')b' + a(s(a) + a'b))$
- $a(s(a')b' + (as)(a'b'))$

This is no longer true in a non-commutative case.

Lemma VII 3.1:

Let $s$ be a $(s, t)$-derivation of $A$ and $a$ be an element of $A$. If there exist algebra automorphisms $s'$ and $t'$ of $A$ such that $as't = as$ and $at't = at$, then the linear endomorphism $as$ is a $(s', t')$-derivation and $as$ is a $(s, t')$-derivation.

Proof: For any $b, b' \in A$,

$(as)(bb') = a(s(bb'))$

- $a(s(b)b) + a(s(b)t(b'))$

$a(s = as't$ $\Rightarrow$ $= s(s(b) + a(s(b)t(b))$

- $s(b)(as)(b) + (as)(b)t(b')$

Thus, $as$ is a $(s', t')$-derivation.

Similarly, $(as)(bb') = a(s(bb'))$

- $s(b)(as)(b) + (as)(b)t(b')$

- $a(s)(bb')$

$a(t = at't$ $\Rightarrow$ $= s(b)(as)(b) + s(b)at'(b)$

- $s(b)(as)(b) + (as)(b)t(b')$

$\checkmark$
We now turn to the quantum plane $A = k_q[x, y] = k[x^\mathbb{Z}, y^\mathbb{Z}]/I_q$, where $k$ is the ground field, $I_q$ is the two-sided ideal of the free algebra $k[x, y]$ generated by $x$ the element $yx - qxy$.

Let us consider its algebra automorphisms $\sigma_x$ and $\sigma_y$ defined by

\[
\sigma_x(x) = qx, \quad \sigma_x(y) = y, \quad \sigma_y(x) = x, \quad \sigma_y(y) = qy.
\]

Note that when $q = 1$, we have $\sigma_x = \sigma_y = id$, i.e., $k[x, y]$.

We define $q$-analogues $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ of the classical partial derivatives by

\[
\frac{\partial}{\partial x}(x^m y^n) = [m] x^{m-1} y^n \quad \text{and} \quad \frac{\partial}{\partial y}(x^m y^n) = [n] x^m y^{n-1},
\]

where $[n] = q^n - q^{-n}$.

**Proposition VII 3.2.**

(a) $x$, $y$, $xy$, $yx$, $\sigma_x$, $\sigma_y$, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ have the following relations:

\[
\begin{align*}
\sigma_x x &= qx y, & x y &= q y x, \\
\sigma_x x y &= q x y, & x y \sigma_y &= q y x \sigma_y, \\
\frac{\partial}{\partial x} y &= q y \frac{\partial}{\partial x} x, & \frac{\partial}{\partial y} x &= q x \frac{\partial}{\partial y} y, \\
\frac{\partial}{\partial x} x y &= q x y \frac{\partial}{\partial x} x + \sigma_x = q x y \frac{\partial}{\partial x} x + \sigma_x, \\
\frac{\partial}{\partial y} x y &= q y \frac{\partial}{\partial y} x + \sigma_y = q y \frac{\partial}{\partial y} x + \sigma_y.
\end{align*}
\]

Let

\[
\begin{align*}
x \frac{\partial}{\partial x} &= \frac{\sigma_x - \sigma_x^{-1}}{q - q^{-1}}, & y \frac{\partial}{\partial y} &= \frac{\sigma_y - \sigma_y^{-1}}{q - q^{-1}}.
\end{align*}
\]
(b) The endomorphism $\frac{\partial}{\partial x}$ is a $(\partial_x, \partial_y, \partial_x)\text{-}\text{derivation}$

and similarly $\frac{\partial}{\partial y}$ is a $(\partial_y, \partial_x\partial_y)\text{-}\text{derivation}$.

\textbf{Proof of (a)}: 

By Prop. IV, 1.1 (b), we know that $y^i x^i = q^i x^i y^i$

so we only need to verify the relations for $x^i y^j$.

1. $(y^i x^j)(x^i y^j) = y^i x^i y^j$
   
   $= q^i x^i y^i y^j$
   
   $= q^i y^i x^i y^j$
   
   $= q x (q^i x^i y^j) y^j$
   
   $= q x (y x^i y^j) y^j$
   
   $= q x (x y^j) y^j = [(x x^i y^j)(x^i y^j)]$

2. $(\partial_x x^j)(x^i y^j) = \partial_x (x^i y^j)$
   
   $= q^i x^i y^j$
   
   $= q x (q^i x^i y^j)$
   
   $= q x (\partial_x)(x^i y^j)$

3. $(\frac{\partial}{\partial x} \partial_x)(x^i y^j) = \frac{\partial}{\partial x} (\partial_x (x^i y^j))$
   
   $= \frac{\partial}{\partial x} (q^i x^i y^j)$
   
   $= q^i (x x^i y^j)$
   
   $= q^i (\partial_x)(x^i y^j)$
   
   $= q^i (\frac{\partial}{\partial x} (x^i y^j))$
\( \frac{\partial^2}{\partial x^2} \left( f(x, y) \right) = \frac{\partial^2}{\partial x^2} \left( f(x, y) \right) \)

\( \frac{\partial^2}{\partial y^2} \left( f(x, y) \right) = \frac{\partial^2}{\partial y^2} \left( f(x, y) \right) \)

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$a$ is a

(6.9 Pf of (b). Recall that $\sigma = (\sigma', \tau)$ - derivation if

$$\sigma(a') = \sigma(a) \sigma'(a') + \sigma'(a) \tau(a'),$$

and it's equivalent to

$$\sigma a = \sigma(a) \sigma + \sigma'(a) \tau.$$ (Equation 6.9)

In addition, we note that relation (8) holds for $a, a'$, then it holds for $aa'$.

Check: $\sigma(aa') = \sigma(a) \sigma(a')$

$$= \sigma(a) \sigma + \sigma'(a) \tau.$$ (Equation 6.9)

$$\sigma a' = \sigma(a) \sigma + \sigma'(a) \tau$$

$$\sigma a = \sigma(a) \sigma + \sigma'(a) \tau.$$

$$\tau a = \tau(a) \tau + \sigma'(a) \tau$$

$$\sigma(a') = \sigma(a) \sigma + \sigma'(a) \tau$$

We only need to check relation (6x) for $a=x$ and $a=y$ when $\sigma = \frac{\partial x}{\partial z}, \quad \sigma = \frac{\partial x}{\partial y}, \quad \tau = \frac{\partial x}{\partial z}.$

Check:

$$\sigma x y = \frac{\partial x}{\partial z} y + \frac{\partial y}{\partial z} (x) \frac{\partial y}{\partial z} = \frac{\partial x}{\partial z} y + \sigma x \quad \text{by (5)}.$$

Therefore, relation (6x) holds for $a=x$ & $a=y$ when

$$\sigma = \frac{\partial x}{\partial z}, \quad \sigma = \frac{\partial x}{\partial y}, \quad \tau = \frac{\partial x}{\partial z}.$$ Hence, $\frac{\partial y}{\partial z}$ is a $(\sigma x y, \frac{\partial x}{\partial z})$ - derivation.

Similarly, we can show that $\frac{\partial y}{\partial z}$ is a derivation of $(\sigma y, \frac{\partial x}{\partial z})$ - derivation. We omit the details.
The following theorem shows that $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ endow the quantum plane with the structure of a module algebra over the Hopf algebra $U_q^h$.

**Theorem VII 3.3.** For any $P \in \mathbb{k}[x, y]$, set

$$ZP = x \frac{\partial}{\partial y} P, \quad ZP = \frac{\partial}{\partial x} P, y$$

$$K^+ P = (5 \cdot 5^+)^{-1} (P), \quad K^- P = (5 \cdot 5^-)^{-1} (P)$$

(a) Formulas $(Z, Z, K^+, K^-)$ define the structure of a $U_q^h$-module algebra on $\mathbb{k}[x, y]$.

(b) The subspace $\mathcal{H}^0$ of homogeneous elements of degree 1 in $\mathcal{H}$ is a $U_q^h$-submodule of the quantum plane. It is generated by the highest weight vector $x^0$ and is isomorphic to the simple module $V_{0,0}$.

**Proof:** Theorem VII 3.3 is the quantum version of Theorem VI 6.4. It follows that the quantum plane contains all finite-dimensional simple $U_q^h$ modules.

**Proof of (b):** We first show that the formulas $Z, Z, K^+, K^-$ equip
**Proof of (a):**

We first show that the formulas (i) equip $k[x,y]$ with a $U_q$-module structure, so we have to check Relations 1.1.19.

For 1.1.9, $KK^{-1} = 5x_5y^{-1} x_5y^{-1} = 1$.

For 1.1.10, $k^{-1} = y_5 x_5 y_5^{-1} = 1$.

For 1.1.11, $KEK^{-1} = 5x_5y^{-1} x_5y^{-1} 5x_5y^{-1}$

\[
= 5x_5y^{-1} x_5y^{-1} 5x_5y^{-1} \]

by def. of $5y$.

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= 5x_5y^{-1} x_5y^{-1} 5x_5y^{-1} \]

by $1$.

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by $3$.

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= q^2 x_5y^{-1} x_5y^{-1} 5x_5y^{-1} \]

by def. of $5y$.

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by $1$.

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by def. of $5y$.

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= q^2 x_5y^{-1} x_5y^{-1} 5x_5y^{-1} \]

by $1$.
For any pair of \( P, Q \) of elements of \( \mathbb{R} \):

1. \( P \neq Q \Rightarrow \mathbb{P}(P) = \mathbb{P}(Q) \)

2. \( E(P) = E(Q) \)

3. \( K(P) = E(P) \cdot K(Q) \)

4. \( K(Q) = Q \cdot K(P) \)

Next, we have

By Lemma 1.2, it's enough to check for any \( Q \).

We now prove that the quotient plane is an \( R \)-algebra:

\[
\frac{\mathbb{R}[x] - \mathbb{R}[x]}{\mathbb{R}[x] - \mathbb{R}[x]} = R
\]

\[
\frac{\mathbb{R}[x] - \mathbb{R}[x]}{\mathbb{R}[x] - \mathbb{R}[x]} = R
\]

\[
\frac{\mathbb{R}[x] - \mathbb{R}[x]}{\mathbb{R}[x] - \mathbb{R}[x]} = R
\]
\( \phi(E) = \phi(F) = 0, \phi(K) = \phi(K^{-1}) = 1 \)

1. follows from VI 1 (1x3) and the definitions of \( \Phi_x, \Phi_y, \frac{\Phi_y}{\Phi_x}, E, F, \phi_x, \phi_y \), and \( K \).

2. follows from the definition of \( K \) and that \( \phi_x \) and \( \phi_y \) are algebra automorphisms.

3. By Lemma 3.1 and Prop 3.2 (b), we note that \( \phi_y \phi_x \) is a \((\phi_x \phi_y, \phi_x)\)-derivation and \( \phi_x \phi_y \) is \((\phi_x \phi_y, \phi_y)\)-derivation. These together with the definitions of \( E, F, K \) \& \( \phi_x \phi_y \) imply (3) & (6).

This completes the proof of (6).

**Theorem 7.2.**

**Proof of (b).**

By the definitions of \( E, F, K \), it's easy to show that:

\[ E x^n = x \frac{\partial}{\partial y}(x^n) = 0 \]
\[ K x^n = \phi_y \phi_x^{-1}(x^n) = \phi_y(x^n) = q^n x^n \]

Thus, \( x^n \) is a highest weight vector of weight \( q^n \).

By the definition of \( F \), we have:

\[ \frac{1}{[\phi_x]} F(x^n) = \frac{1}{[\phi_y]} \phi_y^{-1}(x^n) = \left( \frac{E + F}{F + E} \right) x^n = q^n x^n \]

A basis for \( kq^n \mathfrak{g} \mathfrak{n} \) is generated by \( x^n \) under action of \( F \).

Therefore, \( x^n \) generates \( kq^n \mathfrak{g} \mathfrak{n} \).