We can now easily show the standard duality exists.

**Thm 6** For the bialgebra

\[ U_q = U_q(sl_2) \quad H = U_q(21) \quad q \text{ not root of } 1 \]

There is unique duality \( \langle , \rangle : U_q \otimes H \to k \)

<table>
<thead>
<tr>
<th>( \langle , \rangle )</th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( f )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( k )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( k^* )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

**pf** Recall a basis for \( H = U_q(21) \):

\[ \lambda_n \otimes u_t, \quad n, t \in \mathbb{N} \quad u_t \text{ eigen.} \]

Define \( e^*, f^*, k^*, (k^*)^* \in H^* \) as follows:

\[ e^* f^* k^* (k^*)^* \in H^* \]
\[
\begin{align*}
( e^\nu ( x_1^\alpha d \nu ) )_{\alpha \leq i, i \leq n} &= \begin{pmatrix}
0 & \cdots & 0 \\
0 & \ddots & 0 \\
0 & 0 & 0
\end{pmatrix} \\
( f^\nu ( x_1^\alpha d \nu ) )_{\alpha \leq i, i \leq n} &= \begin{pmatrix}
0 & \cdots & 0 \\
0 & \ddots & 0 \\
0 & 0 & 0
\end{pmatrix} \\
( k^\nu ( x_1^\alpha d \nu ) )_{\alpha \leq i, i \leq n} &= \text{diag} \left( \gamma^{\nu}, \gamma^{\nu \alpha}, \gamma^{\nu \gamma}, \ldots, \gamma^{\nu n} \right) \\
( ( k^\nu )^\nu ( x_1^\alpha d \nu ) )_{\alpha \leq i, i \leq n} &= \text{diag} \left( \gamma^{\nu}, \gamma^{\nu}, \gamma^{\nu}, \ldots, \gamma^{n} \right)
\end{align*}
\]

\[
\text{Recall} \quad [\nu] = \frac{\gamma^{n} - \gamma^{n-\nu}}{\gamma^{n} - \gamma^{\nu}}
\]
In the algebra $H^*$

$e^v, f^v, k^v, (k^v)^* \text{ satisfy the defining relations for } U_q$.

So $\exists$ alg. morphism $\psi: U_q \rightarrow H^*$

That sends

$e \rightarrow e^v, \quad f \rightarrow f^v, \quad k \rightarrow k^v, \quad (k^v)^* \rightarrow (k^v)^*$

Define

$\langle, \rangle: U_q \times H \rightarrow k$

$u \cdot x \rightarrow \psi(u)(x)$

So $\langle, \rangle$ is bilinear.

By construction $\langle, \rangle$ satisfies $\&$.
Since \( \psi \) is an \( \ell^q \) morphism,

\[
\langle u, v, x \rangle = \sum_{x} \langle u, x' \rangle \langle v, x'' \rangle \quad \forall u, v \in \mathcal{U}_q, \quad x \in H
\]

and

\[
\langle 1, x \rangle = \mathcal{E}(x) \quad \forall x \in H
\]

One checks

\[
\langle u, 1 \rangle = \mathcal{E}(u) \quad \forall u \in \mathcal{U}_q
\]

For \( u \in \mathcal{U}_q \) s.t.:

\[
\langle u, xy \rangle = \sum_{x} \langle u, x \rangle \langle u^*, y \rangle \quad \forall x, y \in H
\]

"Condition \( Cl(u) \)"

By const.

\[
C(e), \ C(f1), \ C(k1), \ C(k2) \text{ hold,}
\]

For \( u, v \in \mathcal{U}_q \) s.t. \( Cl(u) \), \( Cl(v) \) hold, s.t. \( Cl(uv) \) holds, \( pf \) is identical to case \( q=1 \), and omitted.

We have shown \( \langle , \rangle \) is a standard duality.

One checks \( \langle , \rangle \) is unique.
Let \( I = \text{z-sided ideal of } \mathcal{M}_q(2) \) gen by \( H \),

\[
\text{ad-}q^\text{thH} = 1
\]

\[
\mathfrak{S}_q(2) \cong \mathcal{M}_q(2)/I
\]

**LEM \( \mathcal{X} \)** For the duality \( \langle \cdot, \cdot \rangle \) in Thm 6,

\[
\langle u_q, I \rangle = 0
\]

pf. \underline{Claim 1}

\[
\langle 1, I \rangle = 0
\]

pf \( \underline{Claim 1} \) \n
\[
\forall x, y \in H
\]

\[
\langle 1, x(ad_q - 1)y \rangle = \varepsilon(x) \varepsilon(ad_q - 1) \varepsilon(y)
\]

\[
= \varepsilon(x) \varepsilon((ad_q - 1) \varepsilon(y)
\]

\[
= 0
\]
\[
\text{Claim 1.2}
\]
\[
\langle e, I \rangle = 0 \quad \langle f, I \rangle = 0
\]
\[
\langle k, I \rangle = 0 \quad \langle k^{-1}, I \rangle = 0
\]
\[
\forall x, \eta \in H
\]
\[
\langle e, x (\text{det}_x)^{-1} \eta \rangle =
\]
\[
\langle e, x \rangle \langle k, \text{det}_y \rangle \langle k^{-1}, \eta \rangle +
\]
\[
\varepsilon (x) \langle e, \text{det}_y \rangle \langle k, \eta \rangle +
\]
\[
\varepsilon (x) \varepsilon (\text{det}_y^{-1}) \langle e, \eta \rangle
\]
\[
\quad = 0
\]
\[
\text{Other cases similarly.}
\]
\[
\text{Claim 3}
\]
\[
\Delta (I) \leq I \otimes H + H \otimes I
\]
\[
\forall x, \eta \in H
\]
\[
\Delta (x (\text{det}_x)^{-1} \eta) = \Delta (x) \underbrace{\Delta (\text{det}_x^{-1}) \Delta (\eta)}_{\text{det}_x \cdot \text{det}_x^{-1} = I}
\]
\[
\quad \leq I \otimes H + H \otimes I
\]
Claim

Given $u, v \in U$, let

\[ \langle u, I \rangle = 0 = \langle v, I \rangle \]

then

\[ \langle uv, I \rangle = 0 \]

Proof

\[ \forall x \in I, \]

\[ \langle uv, x \rangle = \sum_{(x')} \langle u, x' \rangle \langle v, x'' \rangle \]

if

\[ 0 \text{ by } d_3 \]

\[ x' \in I \land x'' \in I \]

\[ = 0 \]

Result follows.
Recall \( I = \text{2-sided ideal of } \mathfrak{m}_q(2) \) generated by \( \text{ad-} q b c I \).

\[ \mathfrak{sl}_q(2) = \mathfrak{m}_q(2)/I \]

By Lemma X and the construction, the duality \( \langle \cdot, \cdot \rangle \) in \( \mathfrak{m}_q(2) \) induces a duality \( \langle \cdot, \cdot \rangle : \mathcal{U}_q \times \mathfrak{sl}_q(2) \rightarrow k \).

Recall \( \mathfrak{sl}_q(2) \) is Hopf alg. Its antipode satisfies:

\[ \mathcal{S}(q) = q^{-1} \]

\[ \mathcal{S}(1) = -q b \]

\[ \mathcal{S}(c) = -q^{-1} c \]

\[ \mathcal{S}(t) = a \]

Recall \( \mathcal{U}_q \) is also a Hopf alg. Its antipode satisfies:

\[ \mathcal{S}(e) = -e q^4 \]

\[ \mathcal{S}(f) = -k f \]

\[ \mathcal{S}(k) = k c^{-1} \]

\[ \mathcal{S}(k^2) = k \]
LEM 8 \[ \text{For } u \in U_9 \text{ and } x \in S_L(121) \]

\[ \langle S(u), x \rangle = \langle u, S(x) \rangle \]

*Proof*

One checks that

\[ u \in \{ e, f, k, k^{-1} \}, \quad x \in \{ a, b, c, d \} \]

\[ \begin{aligned}
\varepsilon_{\text{refl}} & \langle S(e), a \rangle = \langle e, S(a) \rangle \\
& = \langle e, d \rangle \\
& - \langle e k^{-1}, a \rangle \\
& + \sum \langle e, a \rangle \langle k^{-1}, a \rangle \\
& + \langle e, b \rangle \langle k^{-1}, c \rangle \\
& = 0
\end{aligned} \]

Given \( u \in \{ e, f, k, k^{-1} \} \) and \( x, y \in S_L(121) \),

\[ \langle S(u), x \rangle = \langle u, S(x) \rangle \]

\[ \langle S(u), y \rangle = \langle u, S(y) \rangle \]

Show

\[ \langle S(u), x y \rangle = \langle u, S(x y) \rangle \]
\[
\langle s(e), x \eta \rangle = -\langle e k^\nu, x \eta \rangle
\]
\[
\Delta(e k^\nu) = \Delta(e) \Delta(k)^\nu = (e^2 \omega + i e k^\nu)(k^\nu e k^\nu) = e k^\nu \omega + k^\nu e k^\nu
\]
\[
= -\langle e k^\nu, x \eta \rangle \langle 1, \eta \rangle - \langle i c^\nu, x \eta \rangle \langle e k^\nu, \eta \rangle
\]
\[
\text{OK}
\]
Case \( n = 1 \)

\[
\langle s(f), x \eta \rangle = \langle f, s(x \eta) \rangle
\]

\[
\langle f, s(x \eta) \rangle = \langle f, s(y) s(x) \rangle
\]

\[
= \sum_{(a)} \langle f', s(y) \rangle \langle f'', s(x) \rangle
\]

\[
\Delta(f) = k \cdot f + f \otimes 1
\]

\[
= \langle k^\times, s(y) \rangle \langle f, s(x) \rangle + \langle f, s(y) \rangle \langle 1, s(x) \rangle
\]

\[
= \langle k^\times, y \rangle \langle f, x \rangle + \langle s(f) \rangle \langle s(y) \rangle \langle s(x) \rangle
\]

\[
\langle k, \eta \rangle = -\langle k f, x \rangle
\]

\[
\Delta(k f) = \Delta(k) \Delta(f)
\]

\[
= (k \otimes k)(k \cdot f + f \otimes 1)
\]

\[
= 1 \otimes k f + k f \otimes k
\]

\[
= -\langle 1, x \rangle \langle k f, \eta \rangle
\]

\[
\Delta(\langle k f, x \rangle \langle k, \eta \rangle)
\]

\[
= 0 \otimes k
\]
Case $u \neq k$

$\langle s(k), x \rangle = \langle k, s(x) \rangle$

$$\langle \Delta(k^2) x, x \rangle = k^2 \delta(x)$$

$\langle k^2 x, \omega \rangle \langle k^2 \omega, \omega \rangle$

$\langle s(k^2), \omega \rangle = \langle k, s(x) \rangle$

$\langle \sigma(k^2), \omega \rangle = \langle k, s(x) \rangle$

$\langle s(\sigma(k^2)), \omega \rangle = \langle s(k^2), \omega \rangle$

$\langle s(k^2), \omega \rangle = \langle k^2 \omega, \omega \rangle$

$\langle k^2 \omega, \omega \rangle = \langle k^2 \omega, \omega \rangle$

$\langle \Delta(k^2) \omega, \omega \rangle = k^2 \delta(\omega)$

So far, $\sigma$ has an $\omega$ in $\{e, f, k, k^2\}$ and $\forall x \in SL_2(\mathbb{C})$

Given $u, v \in SL_2(\mathbb{C})$, set

$\langle s(u), x \rangle = \langle u, s(x) \rangle$ $\forall x \in SL_2(\mathbb{C})$

$\langle s(v), x \rangle = \langle v, s(x) \rangle$

$\langle s(uv), x \rangle = \langle uv, s(x) \rangle$ $\forall x \in SL_2(\mathbb{C})$
\[
\langle s(u), x \rangle = \langle s(u), s(u), x \rangle
\]
\[
= \sum \langle s(u), x' \rangle \langle s(u), x'' \rangle
\]
\[
= \sum \langle u, s(x') \rangle \langle u, s(x'') \rangle
\]

Also,
\[
\langle uv, s(x) \rangle = \sum \langle u, s(x') \rangle \langle v, s(x'') \rangle
\]
\[
\left[ \circ \circ \circ \circ \circ = s \circ o \right]
\]
\[
= \sum \langle u, s(x') \rangle \langle v, s(x'') \rangle
\]

Result follows \qed