Quantum Groups in Statistical Mechanics

Outline:
- What is statistical mechanics? What kinds of problems are studied?
  - Define the 6-vertex model and its partition function.
  - Transfer matrices and R matrices.
  - How quantum groups fit in.

What is statistical mechanics?
- Studies systems of large number of particles (atoms or molecules), using statistical and probabilistic methods.
- Want to relate microscopic behavior to macroscopic properties.
  - Microscopic mechanical laws \( \rightarrow \) temperature, pressure, density, temperature, etc.
  - Examples of problems:
    - What happens on phase changes: e.g. water to ice or freezing.
    - Magnetization of an iron bar.

6-vertex model
- Ice-type model, first introduced by Pauling in 1935.
  - Used to model ice, KDP crystals, ferroelectric, or multiferroic crystals.
- 2-dimensional lattice.

Periodic boundary conditions.

For each edge we associate a variable which can have values + or - (spin).

A configuration of the system is an assignment of spins to each edge. \( 2^M \) different configurations.
Define a probability measure on the set of all configurations. At each vertex $v$ we'll allow 6 possible configurations, each weighted $a, b, c$. 

\[ \begin{array}{c}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4
\end{array} \quad \begin{array}{c}
\varepsilon_5 \\
\varepsilon_6
\end{array} \]


Weight $a$: $+-+-$  
Weight $b$: $-++-$  
Weight $c$: $++--$

\[ R_{\varepsilon_1, \varepsilon_2} - \text{Boltzmann weights.} \]

Weight of configuration $C$ is product of Boltzmann weight over all vertices.

\[ W(C) = \prod_v R_{\varepsilon_1, \varepsilon_2}(v). \]

If we let $Z_{MN} = \sum_C W(C)$, (partition function), then define our probability measure.

by giving probability $\frac{W(C)}{Z_{MN}}$ to configuration $C$.

Turns out that the partition function $Z_{MN}$ is the main object of study. (Somewhat captures the important thermodynamic properties of the system).

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Transfer Matrix + R matrices.

Suppose we look at just one column

\[ \begin{array}{c}
V_N \\
V_{N-1} \\
\vdots \\
V_3 \\
V_2 \\
V_1 \\
V_0 - \text{auxiliary}
\end{array} \quad \begin{array}{c}
\varepsilon_N \\
\varepsilon_{N-1} \\
\vdots \\
\varepsilon_3 \\
\varepsilon_2 \\
\varepsilon_1
\end{array} \]

For each choice of $\varepsilon_1, \ldots, \varepsilon_N$, $\varepsilon'_1, \ldots, \varepsilon'_N$ define

\[ T_{\varepsilon_1, \ldots, \varepsilon_N, \varepsilon'_1, \ldots, \varepsilon'_N} = \sum_{v_{1M} \varepsilon_1, \ldots, v_{NM} \varepsilon_N} R_{v_{1M} \varepsilon_1, v_{2M} \varepsilon_2} \cdots R_{v_{N-1M} \varepsilon_{N-1}, v_{NM} \varepsilon_N}. \]

(can think of this as prob of going from $\varepsilon_1, \ldots, \varepsilon_N$ to $\varepsilon'_1, \ldots, \varepsilon'_N$).

If we associate to each row vector space $V_i = CV_i \oplus CV_i$, then this defines an operator on $V_{1N} \otimes T$.

This is the column transfer matrix.
There are $M$ columns, let $e_i = (e_{i1}, e_{i2}, \ldots, e_{in})$ be the configuration of edges between columns $i$ and $i+1$. Then
\[
Z_{M,N} = \sum_{e_{i1}} \sum_{e_{i2}} \cdots \sum_{e_{in}} T_{e_{i1}} T_{e_{i2}} \cdots T_{e_{in}}
\]
\[
= \text{tr} \left( T^M \right). \quad \text{we want to understand spectrum of } T.
\]
Operator $T$ counts contribution from vertices on a column. If we want to look at contribution from a single vertex, we define another operator $R \in \text{End} \left(V \otimes V\right)$.
\[
R (v_i \otimes v_{i'}) = \sum_{e_{i1}, e_{i2}} v_i \otimes v_{i'} R_{e_{i1}, e_{i2}}
\]
In basis $\{v_i \otimes v_i, v_i \otimes v_{i'}, v_{i'} \otimes v_i, v_{i'} \otimes v_{i'}\}$
\[
R = \begin{bmatrix}
    a & b & c \\
    b & c & a \\
    c & a & b
\end{bmatrix}
\]
It can then define $R_{ij}$ on $V_i \otimes \cdots \otimes V_n$ which acts as $R$ on $V_i$ and $V_j$ and the identity for all $V_k \neq V_i, V_j$.

- Relation between $R$ and $T$: Monodromy matrix $\rightarrow$ column
If we identify vertex $v_0 \in Cx_1 \oplus Cx_2$, then we can define the monodromy matrix
\[
T = R_{01} R_{02} \cdots R_{0n} \quad \text{acts on } V_0 \otimes V_1 \otimes \cdots \otimes V_n.
\]
Transfer matrix $T = \text{tr}_{V_0} (T)$.
Relation to Quantum Groups

Understand T by Bethe Ansatz. (a way to diagonalize T).

Recall that the model depends on Bethemann weights \( a, b, c \).

When do transfer matrices of different parameters \( T(a, b, c) \), \( T(a', b', c') \) commute?

\[
\Delta = \frac{a^2 + b^2 - c^2}{ab} \quad \text{the anisotropy parameter.}
\]

when \( \Delta(a, b, c) = \Delta(a', b', c') \).

Furthermore, scaling \( a, b, c \) by \( p \) scales partition function.

\[
Z_{MN}(p a, p b, p c) = p^{MN} Z(a, b, c).
\]

Fixing scale and fixing \( \Delta \), we can parametrize weights:

- one way to do so is the following:
  \[
  \begin{align*}
  a &= p \sin(u + \nu) \\
  b &= p \sin(u) \\
  c &= p \sin(\nu).
  \end{align*}
  \]

\( p \) fixes scale, \( u \) fixes \( \Delta \).

Commutativity of \( T(u) \) and \( T(u') \) follows from the fact that \( R(u) \) satisfies

Yang-Baxter equation:

(see Eben Dem Weng) rules

\[
R_{12}(\xi_{1}/\xi_{2}) R_{13}(\xi_{3}/\xi_{1}) R_{23}(\xi_{2}/\xi_{3})
\]

\[
= R_{23}(\xi_{2}/\xi_{3}) R_{13}(\xi_{3}/\xi_{1}) R_{12}(\xi_{1}/\xi_{2}).
\]

\( R \) matrices can be seen as intertwining maps for modules over \( U_q(\mathfrak{sl}_2) \).

(affine quantum enveloping algebra of \( \mathfrak{sl}_2 \)).

Given a solution to Yang-Baxter equation, we can get a solvable model, can build or solve other models this way. So study of representations of \( U_q(\mathfrak{sl}_2) \) becomes important for this.