Basic definition
Def: For a group G, a representation is a homomorphism \( \rho : G \to GL(V) \) for some vector space \( V \) over a field. We suppose \( V \) is finite-dimensional over \( \mathbb{C} \) here.

Notice that: a representation of \( G \) is the same as a \( \mathbb{C} \)-algebra (left) \( \mathbb{C} \)-module \( V \).

Here, if \( S \) is a set, then \( \mathbb{C}S \cong \mathbb{C}[-S] \) denotes the free \( \mathbb{C} \)-module with basis \( S \).

\( \mathbb{C}G \) is the group algebra of \( G \) over \( \mathbb{C} \).

Def: A \( \mathbb{C}G \)-module \( V \) is completely determined up to isomorphism by its character \( \chi_V : G \to \mathbb{C} \)
\[ \chi_V(g) = \text{trace}(g : V \to V) \]

The character \( \chi_V \) is a class function, meaning it is constant on \( G \)-conjugacy classes.

The space \( R_\mathbb{C}(G) \) of class functions \( G \to \mathbb{C} \) has a Hermitian, positive definite form
\[ \langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)} \]

Schur's Lemma: two \( \mathbb{C} \)-modules \( V_1, V_2 \) are isomorphic if and only if \( \text{Hom}_G(V_1, V_2) \neq 0 \).

For any two \( \mathbb{C}G \)-modules \( V_1, V_2 \), \( \langle \chi_{V_1}, \chi_{V_2} \rangle = \dim \text{Hom}_G(V_1, V_2) \)

The set of all irreducible characters \( \text{Irr}(G) \) forms an orthonormal basis of \( R_\mathbb{C}(G) \) with respect to this form, and spans a \( \mathbb{Z} \)-sublattice \( R(G) \leq \mathbb{Z} \cdot \text{Irr}(G) \leq R_\mathbb{C}(G) \) sometimes called the virtual characters of \( G \).

For every \( \mathbb{C}G \)-module \( V \), the character \( \chi_V \) belongs to \( R(G) \).

Def: Define a \( \mathbb{C} \)-bilinear form \( \langle , \rangle_\mathbb{C} \) on \( R_\mathbb{C}(G) \) by
\[ \langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)} \]

\( \langle , \rangle_\mathbb{C} \neq \langle , \rangle_0 \), \( \langle \chi_{V_1}, \chi_{V_2} \rangle = \dim \text{Hom}_G(V_1, V_2) \)

\( \langle , \rangle_\mathbb{C} \) is identical with \( \langle , \rangle_\mathbb{C} \) on \( R(G) \times R(G) \). So we use \( \langle , \rangle_0 \) instead of \( \langle , \rangle_\mathbb{C} \).
4.1.4. Induction and restriction

Def: Given a subgroup \( H \triangleleft G \) and \( CH \)-module \( U \), one can use the fact that \( CG \) is a \((CG, CH)\)-bimodule to form the induced \( CG \)-module.

\[ \text{Ind}_{H}^{G} U \cong CG \otimes_{CH} U \] \[ CG \times CG \otimes_{CH} U \rightarrow CG \otimes_{CH} U. \]

The fact that \( CG \) is free as a \((\text{right})\) \( CH \)-module on basis element \( g \in gCH \),

\[ \times \text{Ind}_{H}^{G} U(g) = \frac{1}{|G|} \sum_{k \in gCH} X_{U}(kg^{-1}) \]

a \( CG \)-module \( V \) is isomorphic to \( \text{Ind}_{H}^{G} U \) for some \( CH \)-module \( U \) iff \( \exists \) an \( H \)-stable subspace \( U \subseteq V \) having the property that \( V = \bigoplus_{g \in CH} U \).

The above construction of a \( CG \)-module \( \text{Ind}_{H}^{G} U \) corresponding to any \( CH \)-module \( U \) is part of a functor \( \text{Ind}_{H}^{G} \) from the category of \( CH \)-modules to the category of \( CG \)-modules, whose functor is called induction.

Def: The restriction operation \( \text{Res}_{H}^{G} : V \rightarrow \text{Res}_{H}^{G} V \) restricts a \( CG \)-module \( V \) to a \( CH \)-module.

Frobenius reciprocity asserts the adjointness between \( \text{Ind}_{H}^{G} \) and \( \text{Res}_{H}^{G} \)

\[ \text{Hom}_{CG}(\text{Ind}_{H}^{G} U, V) \cong \text{Hom}_{CH}(U, \text{Res}_{H}^{G} V) \]

as a special case \((S = A = CG, R = CH, B = U, C = V)\) of the general adjoint associativity

\[ \text{Hom}_{S}(A \otimes_{R} B, C) \cong \text{Hom}_{R}(B, \text{Hom}_{S}(A, C)) \]

for \( S, R \) two rings, \( A \) is an \((S, R)\)-bimodule, \( B \) is a left \( R \)-module, \( C \) is a left \( S \)-module.

Def: When \( H \) is a subgroup of \( G \), the restriction \( \text{Res}_{H}^{G} \) of an \( f \in \text{Rc}(A) \) is defined as

the result of restricting the map \( f : G \rightarrow C \) to \( H \). Then \( \text{Res}_{H}^{G} f \in \text{Rc}(H) \).

So \( \text{Res}_{H}^{G} \) is a \( CG \)-linear map \( \text{Rc}(A) \rightarrow \text{Rc}(H) \).

This map restricts to a \( CH \)-linear map \( A \rightarrow X_{H}(CH) \), since we have \( \text{Res}_{H}^{G} X_{U} = X_{\text{Res}_{H}^{G} U} \) for any \( CG \)-module \( V \).
4.1.6. Inflation and fixed points.

Suppose one has a normal subgroup \( K \triangleleft G \). Given a \( C[G/K] \)-module \( U \), say defined by the homomorphism \( \varphi : G/K \to GL(U) \), the inflation of \( U \) to a \( C[G] \)-module \( \text{Infl}^G U \) is defined by the composite homomorphism \( G \to G/K \to GL(U) \). It has the same underlying space \( U \). \text{Infl}^G U \) is actually a pull back \( U \to C[G] \)-module.

We will later use the fact that when \( H \triangleleft G \) is any other subgroup, one has

\[
\text{Res}^H_G \text{Infl}^G_U = \text{Infl}^H_{H/H\cap K} \text{Res}^G_{H/H\cap K} U
\]

(We regard \( H/H\cap K \) as a subgroup of \( G/K \), since the canonical homomorphism \( H/H\cap K \to G/K \) is injective.)

Ref: \( V^K = \{ v \in V : kv = v \text{ for } k \in K \} \). Inflation turns out to be adjoint to the \( K \)-fixed space construction sending a \( C[G] \)-module \( V \) to the \( C[G/K] \)-module \( V^K \).

Note that \( V^K \) is indeed a \( G \)-stable subspace:

\[ P^G : \forall v \in V^K, g \in G, \; Kg(v) = (g, g^{-1}) \cdot Kg(v) = g \cdot (g^{-1}K \cdot g(v)) = g(v) \in V^K \]

One has the adjointness

\[
\text{Hom}_{C[G]}(\text{Infl}^G_U, V) = \text{Hom}_{C[G/K]}(U, V^K)
\]

We will also need the following formula for the character \( X_v \) in terms of the character \( X_v^K \):

\[
X_v^K(gK) = \frac{1}{|K|} \sum_{k \in K} X_v(gk) = \text{trace } gK : V^K \to V^n
\]

To see this, note that when one has a \( C \)-linear endomorphism \( \varphi \) on a space \( V \) that preserve some \( C \)-subspace \( W \subset V \), if \( \pi : W \to W \) is any idempotent projection onto \( W \), then the trace of the restriction \( \varphi|_W \) is equal to the trace of \( \varphi \circ \pi \) on \( V \).

Applying this to \( W = V^K \) and \( \varphi = gK \), with \( \pi = \frac{1}{|K|} \sum_{k \in K} k : V \to V^K \) we can check \( \pi \) is idempotent projection.

Another way to restate \((4.12)\) is \( X_v^K(gK) = \frac{1}{|K|} \sum_{k \in K} X_v(gk) \) \((4.13)\), equivalent.
We have discussed the inflation on modules.

(Important: $K$-fixed space construction can be also defined on class functions.)

For inflation; Inflation $\text{Infl}^G_{\chi}$ of an $f \in R_C(G/K)$ is defined as the composition

$\chi : G/K \to C \to G/K$. This is a class function of $G$ and thus lies in $R_C(G/K)$.

(Thus, inflation $\text{Infl}^G_{\chi}$ is a $C$-linear map $R_C(G/K) \to R_C(G)$.)

We can check that for every $(C \times G/K)$-module $U$ satisfies $\text{Infl}^G_{\chi} Xu = \chi \text{Infl}^G_{\chi} u$, then $\text{Infl}^G_{\chi}$ restricts to a $C$-linear map $R_C(G/K) \to R_C(G)$.

We can also use (4.12) or (4.13) as inspiration for defining a "$K$-fixed space construction" on class functions.

For every class function $f \in R_C(G)$, we define a class function $f^K \in R_C(G/K)$ by

$f^K(gK) = \frac{1}{|K|} \sum_{k \in K} f(\chi_k gK)$, the map $(\cdot)^K : R_C(G) \to R_C(G/K)$ is $C$-linear

and restricts to a $C$-linear map $R_C(G) \to R_C(G/K)$.

Then we have $X^f \chi^K = (X^f)^K$ for every $C \times G$-module $V$. (Creation of $K$-fixed space between module and class function)

If we take this in (4.11), we obtain $(\text{Infl}^G_{\chi} Xu, Xv) = (Xu, Xv)^K$ for any $(C \times G/K)$-module $U$ and any $C \times G$-module $V$ (since $X^f \chi^K = \text{Infl}^G_{\chi} Xu, Xv^K = (Xv)^K$).

By $C$-linearity, we have $(\text{Infl}^G_{\chi} \alpha, \beta) = (\alpha, \beta)^K$ for any class functions $\alpha \in R_C(G/K)$ and $\beta \in R_C(G)$. 


Lem 4.8. Let $G_1$ and $G_2$ be two groups, and $K_1 \triangleleft G_1$, and $K_2 \triangleleft G_2$ be two respective subgroups.

Let $U_i$ be a $CG_i$-module for each $i \in \{1, 2\}$. Then,

$$\text{(4.15)} \quad (U_1 \otimes U_2)^{K_1 \times K_2} = U_1^{K_1} \otimes U_2^{K_2} \quad \text{(as subspaces of } U_1 \otimes U_2)\text{.}$$

pf: The subgroup $K_1 = K_1 \times 1$ of $G_1 \times G_2$ acts on $U_1 \otimes U_2$.

Its fixed points are $(U_1 \otimes U_2)^{K_1} = U_1^{K_1} \otimes U_2$

Similarly, for $K_2 = 1 \times K_2$ of $G_1 \times G_2$ acts on $U_1 \otimes U_2$, we have $(U_1 \otimes U_2)^{K_2} = U_1 \otimes U_2^{K_2}$

Then we have $(U_1 \otimes U_2)^{K_1 \times K_2} = (U_1 \otimes U_2)^{K_1} \cap (U_1 \otimes U_2)^{K_2}$

$= (U_1^{K_1} \otimes U_2) \cap (U_1 \otimes U_2^{K_2}) = U_1^{K_1} \otimes U_2^{K_2}$

The last equation is true from the fact that if $P, Q$ are subspaces of two vector spaces $U$ and $V$ respectively, then

$$(P \otimes V) \cap (U \otimes Q) = P \otimes Q$$
4.2. Three towers of groups.

Here we consider three towers of groups $G_\kappa = (G_0 < G_1 < G_2 < G_3 < \ldots)$
where either
- $G_\kappa = S_\kappa$, the symmetric group
- $G_\kappa = (S_\kappa)^\Pi$, the wreath product of the symmetric group with some arbitrary finite group $\Pi$.
- $G_\kappa = GL_n(F_\kappa)$, the finite general linear group

Here, $(S_\kappa)^\Pi$ is the semidirect product $S_\kappa \times \Pi^n$ in which $S_\kappa$ acts on $\Pi^n$ via

$\sigma((i_1, \ldots, i_n)) = (\sigma^{-1}(i_1), \ldots, \sigma^{-1}(i_n))$.

For each of the three towers $G_\kappa$, there are embeddings $G_i \times G_j \hookrightarrow G_{ij}$
and we introduce maps $\text{ind}^{ij}_{ij}$ taking $C[G_i \times G_j]$-modules to $C[G_{ij}]$-modules, as well as maps $\text{res}^{ij}_{ij}$ carrying modules in the reverse direction which are adjoint:

$\text{Hom}_{C[G_{ij}]}(\text{ind}^{ij}_{ij} U, V) = \text{Hom}_{C[G_i \times G_j]}(U, \text{res}^{ij}_{ij} V)$

Def 4.18. For $G_\kappa = S_\kappa$, one embeds $G_i \times G_j$ into $G_{ij}$ as the permutations as the permutations that permute $\{1, 2, \ldots, i, i+1, i+2, \ldots, i+j\}$ separately.

Here one defines $\text{ind}^{ij}_{ij} U \equiv \text{Ind}_{G_i \times G_j}^{S_\kappa}$, $\text{res}^{ij}_{ij} V \equiv \text{Res}_{G_i \times G_j}^{S_\kappa}$.

For $G_\kappa = (S_\kappa)^\Pi$, one embeds $G_i \times G_j \times G_{i+j}$ into $G_{ij} \times G_{ij}$ as block monomial matrices whose two diagonal blocks have sizes $i, j$ respectively and define

$\text{ind}^{ij}_{ij} U \equiv \text{Ind}_{G_i \times G_j \times G_{i+j}}^{(S_\kappa)^\Pi}$, $\text{res}^{ij}_{ij} V \equiv \text{Res}_{G_i \times G_j \times G_{i+j}}^{(S_\kappa)^\Pi}$.

For $G_\kappa = GL_n(F_\kappa)$, denote just $GL_n$, one embeds $GL_i \times GL_j$ into $GL_{i+j}$ as block diagonal matrices whose two diagonal block have sizes $i, j$ respectively.

Notice that, we can also introduces as an intermediate the parabolic subgroup $P_{ij}$
consisting of the block upper-triangular matrices of the form $
\begin{pmatrix}
G_i & 0 \\
0 & G_j
\end{pmatrix}$

where $G_i, G_j$ lie in $GL_i, GL_j$, respectively and $L$ in $F_{ij}^{*}$ is arbitrary.

We have a quotient map $P_{ij} \rightarrow GL_i \times GL_j$ whose kernel $K_{ij}$ is the set of matrices

of the form $\begin{pmatrix}
0 & L \\
0 & 0
\end{pmatrix}$ with $L$ again arbitrary. One defines

$\text{ind}^{ij}_{ij} U \equiv \text{Ind}_{P_{ij}}^{GL_i \times GL_j}$, $\text{res}^{ij}_{ij} V \equiv \text{Res}_{P_{ij}}^{GL_i \times GL_j}$.
In the case $G_1 = G_{1,2}$, the operation $\text{ind}_{i,j}^{N_2}$ is sometimes called parabolic induction or Harish-Chandra induction. The operation $\text{res}_{i,j}^{N_2}$ is essentially the $K_{i,j}$-fixed point construction $V \mapsto V_{i,j}$. Via (4.7), (4.11), $\text{res}_{i,j}^{N_2}$ is adjoint to $\text{ind}_{i,j}^{N_2}$.

**Proposition 4.19.** For each of the three towers $G_i$, define a graded $\mathbb{Z}$-module.

$$A \cong A(C_0) = \bigoplus_{n \geq 0} R(CG_i)$$

with a bilinear form $\langle \cdot, \cdot \rangle_A$ whose restriction to $A_n \equiv R(CG_i)$ is the usual form $\langle \cdot, \cdot \rangle_{A_n}$, and set $Z \equiv \bigoplus_{n \geq 0} \text{Irr}(CG_i)$ gives an orthonormal $\mathbb{Z}$-basis.

Notice that $A_0 = \mathbb{Z}$ has its basis element 1 equal to the unique irreducible character of the trivial group $G_0$.

Notice that $A_i \otimes A_j = R(CG_i \times CG_j) \cong R(CG_i \times CG_j)$, then we have candidates for product and coproduct defined by $m : \text{ind}_{i,j}^{N_2} : A_i \otimes A_j \to A_{i+j}$

and $\Delta : \text{res}_{i,j}^{N_2} : A_n \to \bigoplus_{i+j=n} A_i \otimes A_j$.

We first show that $m$ and $\Delta$ are adjoint with respect to the forms $\langle \cdot, \cdot \rangle_A$ and $\langle \cdot, \cdot \rangle_{A_0 \otimes A_0}$.

Suppose $U$, $V$, $W$ are modules over $CG_i$, $CG_j$, $CG_{i+j}$, respectively, then we can write the $(CG_i \times CG_j)$-module $\text{res}_{i,j}^{N_2} W$ as a direct sum $\bigoplus X_k \otimes Y_k$ with $X_k$ being $CG_i$-modules and $Y_k$ being $CG_j$-modules, then we have

$$\text{(4.19)} \quad \text{res}_{i,j}^{N_2} X_k = \bigoplus X_k \otimes Y_k.$$

and $c(m(Xu \otimes XV), XV)A = (\text{ind}_{i,j}^{N_2}(Xu \otimes XV), XV)A = (\text{ind}_{i,j}^{N_2}(Xu \otimes XV), XV)G_{i+j}$

$$= (Xu \otimes XV, \text{res}_{i,j}^{N_2} XV)G_{i+j} = (Xu \otimes XV, \bigoplus X_k \otimes Y_k)G_{i+j}$$

$$= \bigoplus_k (Xu \otimes XV, X_k \otimes Y_k)G_{i+j} = \bigoplus_k (Xu \otimes XV, X_k \otimes Y_k)A \otimes B$$

$$= \bigoplus_k (Xu \otimes XV, X_k \otimes Y_k)A \otimes B = \bigoplus_k (Xu \otimes XV, X_k \otimes Y_k)A \otimes B.$$