Notation

Throughout, we fix the following notation:

$\mathbb{F}$ is a fixed prime power

$GL_n = GL_n(\mathbb{F})$

$A = A(GL) = \bigoplus_{n \geq 0} R(GL_n)$

$\Sigma = \bigcup_{n \geq 0} \text{Irr}(GL_n)$, PSH basis of $A$

$\mathcal{P}$ a set of primitive elements

We embed $GL_i \times GL_j \hookrightarrow GL_{i+j}$ block-diagonally: $(GL_i \circ GL_j)$.

$P_{i,j} = \left\{ \begin{pmatrix} GL_i & \ast \\ 0 & GL_j \end{pmatrix} \right\} \subseteq GL_{i+j}$ (called a parabolic subgroup)

$K_{i,j} = \left\{ \begin{pmatrix} I_i & \ast \\ 0 & I_j \end{pmatrix} \right\}$, so we have exact $1 \rightarrow K_{i,j} \rightarrow P_{i,j} \rightarrow GL_i \times GL_j \rightarrow 1$

We identify $R(GL_i) \otimes R(GL_j) \cong R(GL_i \times GL_j)$ via the isomorphism defined by $\psi \otimes \phi \mapsto \psi \star \phi$, where $(\psi \star \phi)(g,h) = \psi(g) \cdot \phi(h)$.

Define $\text{res}_{i,j} : R(GL_{i+j}) \rightarrow R(GL_i) \otimes R(GL_j)$

by $\text{res}_{i,j} \psi = \left. \left[ \text{Res}_{i,j} \psi \right] \right|_{K_{i,j}}$

$\text{ind}_{i,j} : R(GL_i) \otimes R(GL_j) \rightarrow R(GL_{i+j})$

by $\text{ind}_{i,j}(\psi \otimes \phi) = \text{Ind}_{P_{i,j}}^{GL_{i+j}} \text{Ind}_{GL_i \times GL_j}^{P_{i,j}} (\psi \otimes \phi)$

The bialgebra structure on $A$ is given by

$\psi \star \phi = \text{ind}_{i,j} (\psi \otimes \phi)$ \quad ($\psi \in A_i$, $\phi \in A_j$)

$\Delta \psi = \sum_{z \in \mathcal{P}} \text{res}_{i,j} \psi \quad (\psi \in A_i)$.
General Linear Groups

Let $A = A(\text{GL}) = \bigoplus_{n \geq 0} R(\text{GL}_n)$, we have seen that $A$ is a PSH, with PSH basis $\Sigma = \bigsqcup_{n \geq 0} \text{Irr}(\text{GL}_n)$ consisting of irreducible characters. As usual, $\#$ is primitives in $A$, and we put $\mathcal{C} = \Sigma \cap \#$.

By Theorem 3.1.2, we have a decomposition:

$$A = \bigotimes'_{\mathcal{E}} A(p)$$

Here, $\bigotimes'$ denotes a "restricted tensor product"; it is the group of formal symbols $\bigotimes_{p \in \mathcal{E}} A(p)$, where all but finitely many $A(p)$ are $1$, subject to the usual $\mathbb{Z}$-multilinear relations.

Alternatively, we may take

$$\bigotimes'_{\mathcal{E}} A(p) = \text{dirlim} \bigotimes_{p \in F, F \subseteq \mathcal{E}} A(p),$$

the direct limit being over all finite subsets of $\mathcal{E}$.

$\bigotimes'_{\mathcal{E}} A(p)$ comes with a graded bialgebra structure, and our decomposition is an iso of such.

(It is worth it to note that $\bigotimes'_{\mathcal{E}} A(p)$ is the coproduct of the set $\bigotimes_{p \in \mathcal{E}} A(p)$ in the category of $\mathbb{Z}$-algebras.)

**Def:** $C_n = C \cap \text{Irr}(\text{GL}_n)$ is the set of "cuspidal" representations of $\text{GL}_n = \bigoplus_{n \geq 0} \text{Irr}(\text{GL}_n)$. For $p \in \mathcal{E}$, we write $d(p) = n$ if $p \in C_n$. 
**Fact:** A rep $V$ of $GL_n$ is cuspidal iff $V^{k^i} = 0 \forall k^i \neq n, i > 0$.

**PF:** \[ \text{res}(V^i) = V^i \otimes 1 + 1 \otimes V^i + \sum_{i \neq j} \text{res}_{i,j}(V^i) \]

So $V^i$ primitive iff $\text{res}_{i,i}(V^i) = V^i k^i$ is zero $\forall i$.

Q.E.D.

We want to count $|\text{En}|$ (one important feature here is that $\text{En}$ is $0$ in $V^i$).

This starkly contrasts with the case of symmetric groups/wreath products.

To do this, we first need to count $|\text{Irr}(GL_n)|$.

For each $n \geq 1$, let $F_n = \{ f \in \mathbb{F}_2[x] \mid f \text{ is monic, irreducible, } \deg f = n, f \neq x^i \}$.

$F = \bigcup_{n \geq 1} F_n$

**Prop:** $|\text{Irr}(GL_n)| = \#(\text{conjugacy classes of GL}_n)$

$= \left| \begin{array}{c} \text{functions } z : F \to \text{Par} \\ \sum_{f \in F} (\deg f)|z(f)| = n^2 \end{array} \right|$

**PF:** The first equality is a general fact about representations of finite groups.

The second follows from the observation that conj. classes of $GL_n$ are in bijection with rational canonical forms in $GL_n$.

Equivalently, conj. classes are in bijection with $\mathbb{F}_2[x]$-module structures on $\mathbb{F}_2^n$ in which $x$ is invertible (up to isomorphism).

But by the structure theo for PID's, these are in bijection with

$\{ z : F \to \text{Par} \mid \sum_{f \in F} (\deg f)|z(f)| = n \}$

we exclude $x$ from $F$ so that $x$ will not annihilate anything in $\mathbb{F}_2^n$.

Q.E.D.
Prop (4.46): $|E_n| = |F_n|$. 

\[ \textbf{pf:} \] We do induction on $n$. When $n=1$, noting that all $x \in A_1$ are primitive, we have $|E_1| = |\text{Irr}(G_{\mathbf{L}^1})| = |\text{Irr}(F_{\mathbf{L}^1}^x)|$. But $F_{\mathbf{L}^1}^x$ is abelian, so $|\text{Irr}(F_{\mathbf{L}^1}^x)| = |F_{\mathbf{L}^1}^x| = |F_n|$. 

Now let $n \geq 1$. This is where we use the PSH structure on $A$. Recall the notation $N_{\text{Fin}}^E = \{ f: \mathbf{L} \to N | f \text{ has finite support} \}$. Since the iso $A \cong \bigotimes_{\forall E} A(p)$ respects gradings, we have 

\[ A_n = \bigoplus_{x \in N_{\text{Fin}}^E} \bigotimes_{\forall E} A(p)_{dx} \] 

Here $A(p)_{dx}$ is the $dx$-graded part in grading such that $\deg p = 1$. 

But $A(p) \cong \Lambda$, so $A(p)_{dx}$ has basis parametrized by $\text{Par}(dx)$. Hence, comparing bases on both sides of the above equality, 

\[ |\text{Irr}(G_{\mathbf{L}^n})| = \left| \left\{ \text{functions } \xi: \mathbf{L} \to \text{Par} \mid \sum p \cdot |\text{Par}(p)_{dx}| = n \right\} \right| \]

This gives 

\[ |E_n| = |\text{Irr}(G_{\mathbf{L}^n})| - \left| \left\{ \xi: \mathbf{L} \to \text{Par} \mid \sum p \cdot |\text{Par}(p)_{dx}| = n \right\} \right| \]

Similarly, last prop gives 

\[ |F_n| = |\text{Irr}(G_{\mathbf{L}^n})| - \left| \left\{ \xi: \mathbf{L} \to \text{Par} \mid \sum p \cdot |\text{Par}(p)_{dx}| = n \right\} \right| \]

By induction hypothesis, $|\xi| = |F_n|$ for $\forall \xi \in N_{\text{Fin}}^E$, so our prop follows. \[ \square \]
Now how do we write down $|\mathcal{F}_n|$?

As shown above, $|\mathcal{C}_n| = |\mathcal{F}_n| = 2^n - 1$.

When $n > 1$, $\mathcal{F}_n$ is simply all monic irreducibles in $\mathbb{F}_2[x]$ of degree $n$.

By elementary field theory, $2^n = |\mathbb{F}_{2^n}| = \sum_{d|n} d \cdot \text{id}$,

where $\text{id} = |\mathbb{F}_2|$, unless $d = 1$, when $\text{id} = 2$, and the sum is over all positive divisors of $n$.

But then the M"obius inversion formula says

$$n|\mathbb{F}_n| = \sum_{d|n} \mu(d) 2^d,$$

where $\mu(m) = \begin{cases} 0 & \text{if } m \text{ is not squarefree} \\ (-1)^k & \text{if } m \text{ has exactly } k \text{ prime factors} \end{cases}$

Thus:

**Prop:** $|\mathcal{C}_n| = |\mathcal{F}_n| = 2^n - 1$, and for $n \geq 2$, $|\mathcal{C}_n| = |\mathcal{F}_n| = \frac{n}{\phi(n)} \sum_{d|n} \mu(d) 2^d$.

(Note that for $n$ prime, this is simple: $|\mathcal{C}_n| = |\mathcal{F}_n| = \frac{n}{\phi(n)} (2^n - 2).$)

**Example:** Take $g = 2$. Then have the following table for the number of cuspidal representations of $GL_n(\mathbb{F}_2)$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>\mathcal{C}_n</td>
<td>$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td>18</td>
<td>30</td>
</tr>
</tbody>
</table>

**Cor:** $|\mathcal{C}_n| \approx \frac{2^n}{n}$ as $n \to \infty$. (Or as $g \to \infty$)

**Pr:** Since the number of divisors of $n$ is at most $n$, the last prop gives

$$\frac{1}{n} (2^n - 2^{n/2}) \leq |\mathcal{C}_n| \leq \frac{1}{n} (2^n + n 2^{n/2}) \quad \forall n \geq 2,$$

hence $1 - n 2^{-n/2} \leq |\mathcal{C}_n| \leq 1 + n 2^{-n/2}$.

QED.
Unipotent Characters

We will need to use exercise 4.31(d) here:

Let \( x = (x_1, \ldots, x_n) \in \mathbb{N}^n \). We put \( G_x = GL_{x_1}\times \ldots \times GL_{x_n} \).

If \( n = \sum x_i \), then we have embedding \( G_x \hookrightarrow GL_n \) as \( \begin{pmatrix} GL_{x_1} & \circ & \circ \\ \circ & \circ & GL_{x_n} \end{pmatrix} \).

We also define the parabolic subgroup \( P_x = \left\{ \begin{pmatrix} GL_{x_1} & \circ \\ \circ & 0 \end{pmatrix} \right\} \subset GL_n \).

We have natural projection \( P_x \to G_x \).

Then we define \( \text{ind}^n_x : R(GL_{x_1}) \otimes \ldots \otimes R(GL_{x_n}) \to R(GL_n) \)

by \( \text{ind}^n_x (\varphi_1 \otimes \ldots \otimes \varphi_n) = \text{Ind}_{P_x}^{GL_n} \text{Ind}_{G_x}^{P_x} (\varphi_1 \otimes \ldots \otimes \varphi_n) \).

Exercise 4.31(d): \( \text{ind}^n_x (\varphi_1 \otimes \ldots \otimes \varphi_n) = \varphi_1 \cdots \varphi_n \), where the right is multiplication in \( A \).

(One way to prove this is to note that the left is adjoint to the map \( \text{res}^n_x : R(GL_n) \to R(GL_{x_1}) \otimes \cdots \otimes R(GL_{x_n}) \), (see the same exercise for def of res) while the right is \( \text{ind}_{\text{ad}_{x_1}}^{\text{ad}_{x_1} + \ldots + \text{ad}_{x_n}} \circ \left( \otimes \text{ind}_{\text{ad}_{x_1}}^{\text{ad}_{x_1} + \ldots + \text{ad}_{x_n}} \circ \cdots \circ \left( \otimes \text{ind}_{\text{ad}_{x_1}}^{\text{ad}_{x_1} + \ldots + \text{ad}_{x_n}} \right) \right) \circ \left( \otimes \text{res}_{\text{ad}_{x_1}}^{\text{ad}_{x_1} + \ldots + \text{ad}_{x_n}} \right) \circ \cdots \circ \left( \otimes \text{res}_{\text{ad}_{x_1}}^{\text{ad}_{x_1} + \ldots + \text{ad}_{x_n}} \right) \),

which is adjoint to \( \left( \otimes \text{res}_{\text{ad}_{x_1}}^{\text{ad}_{x_1} + \ldots + \text{ad}_{x_n}} \right) \circ \cdots \circ \left( \otimes \text{res}_{\text{ad}_{x_1}}^{\text{ad}_{x_1} + \ldots + \text{ad}_{x_n}} \right) \circ \text{res}_{\text{ad}_{x_1}}^{\text{ad}_{x_1} + \ldots + \text{ad}_{x_n}} \).

By an argument used to show \( A \) is coassociative, one shows \( \text{res}^n_x = \left( \otimes \text{res}_{\text{ad}_{x_1}}^{\text{ad}_{x_1} + \ldots + \text{ad}_{x_n}} \right) \circ \cdots \circ \text{res}_{\text{ad}_{x_1}}^{\text{ad}_{x_1} + \ldots + \text{ad}_{x_n}} \).)

For each \( n \) let \( 1_{GL_n} \) denote the trivial representation of \( GL_n \). Note that \( 1_{GL_1} = 1_{GL_1} \) is cuspidal, so that \( A(1_{GL_1}) \) appears as a factor in our decomposition \( A = \otimes_{\mathbb{R}^+} A(p) \).
**Def.** The unipotent characters of $GL_n$ are precisely those $\chi \in \text{Irr}(GL_n)$ such that $\langle \chi, (1_{GL_n})^n \rangle \neq 0$; i.e., they are the elements of the PSH basis of $A(1_{GL_n})$, in degree $n$.

**Fact.** (1) $(1_{GL_n})^n$ is the character of the representation $C[GL_n/B]$ of $GL_n$, where $B$ is the subgroup of upper triangular matrices.

(2) The unipotent representations of $GL_n$ are the irreducible subrepresentations of $C[GL_n/B]$. Equivalently, they are the irreducible representations $V$ of $GL_n$ such that $V_B \neq 0$.

**Proof:** (1) By exercise 4.31(d), $(1_{GL_n})^n = \text{Ind}_{B}^{GL_n} \text{Ind}_{Q_1}^{GL_n} (1_{Q_1} \otimes 1_{GL_1})$ 
\[ = \text{Ind}_{B}^{GL_n} (1_B) \]
\[ = C[GL_n] \otimes C = C[GL_n/B] \]

(2) $V$ appears as a subrep of $C[GL_n/B]$ iff $\langle \chi_V, (1_{GL_n})^n \rangle \neq 0$.
But by adjointness of $\text{Ind}$ and $\text{Res}$, we have

$\langle \chi_V, \text{Ind}_{B}^{GL_n} (1_B) \rangle = \langle \text{Res}_{B}^{GL_n} \chi_V, 1_B \rangle$, and

the right is nonzero iff $1_B$ is a subrep of $\text{Res}_B^{GL_n} V$. \[\text{QED.}\]

Note that $GL_n/B$ is the variety of complete flags in $\mathbb{P}_F^n$, and the rep $C[GL_n/B]$ corresponds to the canonical action of $GL_n$ on such. In particular, when $n=2$

$GL_2/B = \mathbb{P}^1(F_2)$, the projective line.

**Proposition:** We can choose the PSH isomorphism $\Lambda \cong A(1_{GL_n})$ so that

$h_n \mapsto 1_{GL_n}$. 


**Pf:** By thm (0.18), \( \chi_{\mathbb{C}[GL_2 \mathbb{B}]} = (1_{GL_2})^2 \) is a sum of two irreducible characters. One is \( 1_{GL_2} : \sum_{c \in GL_2 \mathbb{B}} \) is \( GL_2 \)-invariant.

Denote the other by \( St_2 \), so \( (1_{GL_2})^2 = 1_{GL_2} + St_2 \).

To prove our prop., by thm's (0.18) and (0.20) it will be sufficient to show that \( St_2 \cdot 1_{GL_n} = 0 \) for all \( n \), but

\[
\Delta(1_{GL_n}) = \sum_{i,j} \text{res}_{i,j}(1_{GL_n}) = \sum_{i,j} 1_{GL_2} \cdot 1_{GL_2}, \quad \text{so}
\]

\[
St_2 \cdot 1_{GL_n} = \sum_{i,j} \langle St_2, 1_{GL_2} \rangle 1_{GL_2} = \langle St_2, 1_{GL_2} \rangle 1_{GL_{n-2}} = 0
\]

Since \( St_2, 1_{GL_2} \) are distinct irreps of \( GL_2 \).

QED.

Note that the above iso \( \Lambda \cong A(1_{GL_n}) \) induces bijections

\[
\{ \text{unipotent characters of GL}_n \} \leftrightarrow \text{Par}(\Lambda), \quad \text{via Schur Functions}.
\]

We denote by \( \chi^u \) the unipotent character corresponding to \( 2GL_2 \text{Par}(\Lambda) \).

In particular, \( 1_{GL_n} = \chi^{(u)} \), as \( h_n = S_{(u)} \) in \( \Lambda \).

(Generalizing the notation \( St_2 \), we denote \( St_n = \chi^{(u)} \), the image of \( S_{(u)} \) in \( A(1_{GL_1}) \). The \( St_n \) are called the Steinberg characters)

**Example:** Consider \( GL_2(F_{\ell}) \). By the above proof, \( GL_2 \) has exactly two unipotent characters, \( 1_{GL_2} \) and \( St_2 \), and \( 1_{GL_2} + St_2 = \chi_{\mathbb{C}[GL_2 \mathbb{B}]} \).

As noted earlier, \( GL_2 / B = P^1(F_{\ell}) \), so

\[
\chi_{\mathbb{C}[P^1]}(g) = \#(\text{points of } P^1(F_{\ell}) \text{ fixed by } g),
\]

\[
St_2(g) = \#(\text{points of } P^1(F_{\ell}) \text{ fixed by } g) - 1.
\]

In particular, \( \dim St_2 = St_2(F_{\ell}) = |P^1(F_{\ell})| - 1 = \ell \).
Now consider $\text{GL}_2(\mathbb{F}_2)$. One checks that the action of $\text{GL}_2$ on $\mathbb{P}^1(\mathbb{F}_2) = \{(0,1), (1,0), (1,1)\}$ sets up an isomorphism $\text{GL}_2(\mathbb{F}_2) \cong S_3$. The conjugacy classes are then

<table>
<thead>
<tr>
<th>1-cycle</th>
<th>2-cycles</th>
<th>3-cycles</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0,1)$</td>
<td>$(0,1), (1,0), (1,1)$</td>
<td></td>
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<tr>
<td>$(0,1)$</td>
<td>$(0,1)$</td>
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<tr>
<td>$(0,1)$</td>
<td>$(0,1)$</td>
<td></td>
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</tbody>
</table>

Hence, besides $1_{\text{GL}_2}, S_{12}$, there is one more irreducible character: it is given by the sign character $\text{Sign}$ of $S_3$. By our earlier calculation, $1_{\text{GL}_2}=1$

So $\text{Sign}$ must be cuspidal (the only rep which is both unipotent and cuspidal is $1_{\text{GL}_2}$, as a fixed point of $B$ is such for any $W$).

We include all this information in the following table:

<table>
<thead>
<tr>
<th></th>
<th>1-cycles</th>
<th>2-cycles</th>
<th>3-cycles</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1_{\text{GL}_2} = \chi^{(\alpha)}$</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$S_{12} = \chi^{(\delta)}$</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>$\text{Sign}$</td>
<td></td>
<td>$-1$</td>
<td>$1$</td>
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