Given Hopf algebra \( H \) with antipode \( S \).

Recall
\[
S(ab) = S(b)S(a) \quad a, b \in H
\]
\[
S(1_H) = 1_H
\]

\( \sigma \)
\[
S^2 = S \circ S \quad \text{satisfies}
\]
\[
S^2(ab) = S^2(b)S^2(a)
\]

So
\( S^2 \) is an alg morphism.

\text{LEM} \quad \text{With above notation, assume either}

\((i)\) \quad H is commutative,

\((ii)\) \quad H is cocommutative.

Then
\( S^2 = \text{id} \)

pf \((i)\) \quad \forall a \in H

\[
S(a) \cdot 1_H = \sum_{(a)} a_{1}S(a_{2})
\]

\[
\sum_{(a)} S(a) = \sum_{(a_{1},a_{2})} a_{1}S(a_{2}) = \sum_{(a)} S(a_{1})a_{2}
\]
Apply \( S \):

\[
\varepsilon(\eta_1) \varepsilon(\eta_2) = \sum_{\eta_1} \varepsilon(\eta_2) S(\eta_1)
\]

\[
= \sum_{\eta_1} S(\eta_1) S^2(\eta_2)
\]

by Con

So

\[
0 = \sum_{\eta_1} S(\eta_1) \left( S^2(\eta_2) - \eta_2 \right)
\]

\[
= \left( S \star \left( S^2 - \text{id} \right) \right) (\eta)
\]

So

\[
0 = S \star \left( S^2 - \text{id} \right)
\]

But \( S \) is invertible \( \Rightarrow \) \( S \star \text{id} \) \( \Rightarrow \) \( S \cdot \text{id} \)

\[
0 = S^2 - \text{id}
\]

So

\[
S^2 = \text{id}
\]
(iii) (Similar to part (ii)) \(
A q = \sum_{(4)} a_1 a_2 = \sum_{(4)} a_2 a_1
\)

So \(e_{q} / H = \sum_{(4)} a_1 a_2 = \sum_{(4)} a_2 a_1\)

Apply \(S^2\):

\(e_{q} S / H = \sum_{(4)} S^2 a_1 a_2\)

So

\(0 = \sum_{(4)} \left( S^2 a_1 - a_1 \right) a_2\)

\(= \left( \left( S^2 - i |d| \right) \ast S \right) a_1\)

So

\(0 = \left( S^2 - i |d| \right) \ast S\)

So

\(0 = S^2 - i |d|\)

So

\(S^2 = i |d|\) \(\Box\)
Given a $k$-algebra $A$.

The $k$-module $A$ supports a $k$-algebra $A^{op}$ with mult

\[ ab \ (\text{in } A^{op}) = ba \ (\text{in } A) \]

For $f \in \text{End}(A)$ TFAE:

1. $f(ab) = f(b)f(a)$ for all $a, b \in A$ and $f(1_A) = 1_A$

2. $f : A \to A^{op}$ is a $k$-algebra morphism
Prop For a $k$-module $V$, the tensor algebra $T(V)$ is a Hopf algebra with

$$
\Delta(v) = v \otimes 1 + 1 \otimes v \\
\epsilon(v) = 0 \\
S(v) = -v
$$

$v \in V$

pf Recall $T(V)$ is a bialgebra with above $\Delta, \epsilon$

Consider $S$:

3 $k$-module hom

$$
V \rightarrow T(V)^{\text{op}} \\
v \mapsto -v
$$

This extends to a $k$-alg morphism

$$
S : T(V) \rightarrow T(V)^{\text{op}}
$$

by uniq property of $T(V)$

By claim

$$
S(ab) = S(b) S(a) \\
S(1) = 1 \\
S(v) = -v
$$

$a, b \in T(V) \\
H = T(V) \\
v \in V$
Show \( S \) is an antiprode for \( T(v) \):

By emsh

\[
S : T(v) \rightarrow T(v)
\]

is \( k \)-module hom.

show: \( \forall a \in T(v) \)

\[
\varepsilon(a \mid 1_H) = \sum_{(a_1, a_2)} a_1 S(a_2)
\]  \((*)\)

claim 1 \( (*) \) holds \( \forall a = 1_H \)

\[
\Delta(1_H) = 1_H \otimes 1_H
\]

\( (*) \) becomes

\[
\varepsilon(1_H \mid 1_H) = 1_H \otimes S(1_H)
\]

\( \forall \)

ok

claim 2 \( (*) \) holds \( \forall a = v \in V \)

\[
\Delta(v) = v \otimes 1 + 1 \otimes v
\]

\( (*) \) becomes

\[
\varepsilon(v \mid 1_H) = \sum_{v \in V} v S(1_H \mid + 1_H S(v))
\]

\( \forall v \)

ok
Claim 3

Assume (\( \star \)) holds for \( a, b \in T(\mathbb{V}) \).

Then (\( \star \)) holds for \( a \circ b \).

Recall

\[
\Delta(a \circ b) = \Delta(a) \Delta(b) = \sum \sum a_i b_j \otimes a_2 b_2
\]

(\( \star \)) becomes

\[
\varepsilon(a \circ b) |_H = \sum \sum a_i b_j S(a_2 b_2)
\]

\[
\varepsilon(a) \varepsilon(b) |_H = \sum a_i \left( \sum b_j S(b_2) \right) S(a_2)
\]

\[
\varepsilon(b) \sum a_i S(a_2)
\]

\[
\varepsilon(a) \varepsilon(b) |_H
\]
(*) holds by claims 1-3 and since $V$ generates the algebra $T(V)$.

Similarly we have

$$
\varepsilon(a) M = \sum \varepsilon(a_1) a_2 \quad a \in T(V)
$$

So $\varepsilon$ is antipode that turns the bialgebra $T(V)$ into a Hopf algebra. \qed
LEM. For a $k$-module $V$, the symmetric algebra $\text{Sym}_H^k(V)$ is a Hopf algebra with
\[
\Delta(v) = v \otimes 1 + 1 \otimes v, \quad \epsilon(v) = 0, \\
S(v) = -v
\]

pf. Recall
\[
\text{Sym}_H^k(V) = T(V) / J
\]

2-sided ideal $J$ of $T(V)$ is generated by
\[
u v - v \nu, \quad u, v \in V
\]

Consider antipode of $T(V)$
\[
S : T(V) \rightarrow T(V)
\]

Apply $S$ to $J$:
For $u, v \in V$
\[
S(u v - v u) = S(u)S(v) - S(v)S(u)
\]
\[
= u v - v u
\]
\[
\in J
\]
More generally for $a, b \in T(V)$

$$S(a (uv - vu) b) = S(b) S(uv - vu) S(a) \in J$$

$\in J$

So $S(J) \leq J$

Consider the $k$-module hom

$$T(V) \xrightarrow{S} T(V) \xrightarrow{\text{can}} \text{Sym}(V)$$

$(x)$ sends $J \rightarrow 0$

$(x)$ induces a $k$-module hom

$S : \text{Sym}(V) \rightarrow \text{Sym}(V)$

By constr

$S(ab) = S(b) S(a) = S(a) S(b) \quad a, b \in T(V)$

$S(1_V) = 1_V \quad \forall \in V$

$S(1_H) = 1_H \quad H = S(V)$
$S_0 : \text{Sym}(V) \to \text{Sym}(V)$ is $k$-alg morphism.

Show $S$ is antipode.

$S$ makes these diagrams commute:

\[
\begin{array}{c}
H \otimes H \xrightarrow{1_0 \circ S} H \otimes H \\
\alpha \uparrow \hspace{2cm} \downarrow m \\
H \to H \to H \\
\varepsilon \quad u
\end{array}
\]

\[
\begin{array}{c}
S \otimes 1 \\
1 \otimes H \otimes H \xrightarrow{S \otimes 1} H \otimes H \\
\alpha \uparrow \hspace{2cm} \downarrow m \\
H \to H \to H \\
\varepsilon \quad u
\end{array}
\]

because all maps involved are $k$-alg morphisms and for $v \in V$,

\[
\begin{array}{c}
\text{Vert}(V) \to \text{Vert}(V - \{0\}) \\
\uparrow \hspace{2cm} \downarrow \\
v \to 0 \to 0
\end{array}
\]

\[
\begin{array}{c}
\text{Vert}(V) \to \text{Vert}(V - \{0\}) \\
\uparrow \hspace{2cm} \downarrow \\
v \to 0 \to 0
\end{array}
\]
Prop  For a group $G$

the group algebra $kG$ is a Hopf algebra with

$\Delta(\zeta_g) = \zeta_g \otimes \zeta_g$

$\varepsilon(\zeta_g) = 1
g \in G$

$\zeta(\zeta_g) = \zeta_g$

pf  Recall $kG$ is a bialgebra with above $\Delta, \varepsilon$.

Consider $S$.

Exist k-module Hom

$S : kG \to kG$

$\zeta_g \to \zeta_{g^{-1}}$

Show: For $a \in kG$

$\varepsilon(a) \mathbb{1}_H = \sum_{(a)} a_{(1)} S(a_{(2)})$

WLOG

$a = \zeta_g
g \in G$

$\Delta(\zeta_g) = \zeta_g \otimes \zeta_g$
(\star) \text{ becomes}

\[
\varepsilon(\ell_g)\ 1_H = \sum_{n=1}^{\infty} \ell_n \frac{s(n)}{\ell_g}
\]

\[
\text{OK}
\]

(\star) \text{ is verified}

Similarly one checks

\[
\varepsilon(a)\ 1_H = \sum_{n=1}^{\infty} s(n/a) a_n
\]

\[
\forall a \in KG
\]

So S is an antipode that turns the bialgebra KG into a Hopf algebra.
Prop. Given a connected graded bialgebra

\[ H = \bigoplus_{n \in \mathbb{N}} H_n \]

Then \( H \) is a Hopf algebra, and its antipode \( S \) satisfies

\[ S(H_n) \leq H_0 \quad \forall n \in \mathbb{N} \]

pf. (claim 1) \( S \) is a \( k \)-module homomorphism

\[ S : H \rightarrow H \]

Set both

\[ \varepsilon(a) \cdot H = \sum S(a) a_2 \quad a \in H \quad \forall \]

\[ S(H_n) \leq H_0 \quad n \in \mathbb{N} \]

pf (claim 2) We define

\[ S|_{H_n} : H_n \rightarrow H_n \]

by induction

\[ n = 0 : \quad S(1_H) = 1_H \]

\[ n \geq 1 : \quad \text{Recall from } H_n \quad \varepsilon(\cdot) = 0 \text{ so } \]

\[ 0 = \sum S(a) a_2 \]

\[ (a) \]
Recall

\[ \Delta(n) - 40i - 10a \in \bigoplus_{i=1}^{n-1} H_i \otimes H_{n-i} \]

So \( \mathcal{H} \) becomes

\[ o - S(n) H - S(i) H \in \bigoplus_{i=1}^{n-1} S(H_i) H_{n-i} \]

So

\[ S(n) + a \text{ is an element of } H_n \text{ and } \]

is uniquely determined by the action of \( S \) on \( H_1, H_2, \ldots, H_n \)

\[ \text{Claim 2: } \exists K\text{-module hom} \]

\[ \tilde{S} : H \rightarrow H \]

such that \( \tilde{S} \) is both

\[ \tilde{S}(H_1) = \sum \tilde{S}(a) \quad a \in H \]

\[ \tilde{S}(H_n) \subseteq H_n \quad n \in \mathbb{N} \]

\[ \text{pf cl 2: } \tilde{S}_n \text{ to cl1} \]
Claim 3 \[ S = \tilde{S} \]

Proof of Claim 3

Recall the convolution algebra \( \text{End}(H) \) with multiplication \( \ast \) and identity \( \Pi : H \to H \) \( a \to \varepsilon(a) 1_H \).

By cl 1
\[ \Pi = S \ast \text{id} \]

By cl 2
\[ \Pi = \text{id} \ast \tilde{S} \]

Now
\[ S = S \ast \Pi \]
\[ = S \ast \text{id} \ast \tilde{S} \]
\[ = \Pi \ast \tilde{S} \]
\[ = \tilde{S} \]

We have shown \( S \) is an antipode. Result follows. \( \square \)