Given an algebra $A$ with mult

$m : A \odot A \rightarrow A$
$a \odot b \rightarrow ab$

Recall the opposite algebra $A^{op}$ has mult

$m^{op} : A \odot A \rightarrow A$
$a \odot b \rightarrow ba$

Given a coalg $C$ with comult

$\Delta : C \rightarrow C \odot C$
$c \rightarrow \Sigma c \odot c$

The opposite coalg $C^{op}$ has comult

$\Delta^{op} : C \rightarrow C \odot C$
$c \rightarrow \Sigma c \odot c$
Given a bialgebra $H$ with data $m, \Delta$.

Get four bialgebras with data

\begin{array}{c|c}
 m, \Delta & m^\text{op}, \Delta \\
\hline
 m, \Delta^\text{op} & m^\text{op}, \Delta^\text{op}
\end{array}

"relates"

Next goal:

Suppose $H$ is Hopf alg with antipode $S$.

Are $H$'s relatives also Hopf algebras, and if so, what are their antipodes?
Case $m^\circ \Delta$

Antipode $\tilde{S}$ must satisfy:

$\forall a \in H,$

$$\sum_{a_i} \tilde{S}(a_2) = \delta(a_{\downarrow}) l_H = \sum_{a_i} \tilde{S}(a_1) a_2$$

By prev LEM,

$\tilde{S}$ exists iff $S^\tau$ exists, and in this case

$$\tilde{S} = S^\tau$$
Case $m, \Delta^\text{op}$

**Antipode $S^\nu$ must satisfy:**

$\forall a \in H,$

$$\sum_{a_1} S^\nu(a_2) = \mathcal{E}(a_1 | H) = \sum_{a_1} S^\nu(a_1, a_2) \quad (\circ)$$

Here

$$\sum_{a_1, a_2} = \Delta^\text{op}(a_1) = \sum_{a_1, a_2} a_2 \otimes a_1 \quad (\circ)$$

In terms of $\Delta$, $(\circ)$ becomes

$$\sum_{a_2} S^\nu(a_2) = \mathcal{E}(a_1 | H) = \sum_{a_1} S^\nu(a_1, a_2) a_1 \quad (\circ)$$

By prev. lem, $S^\nu$ exits iff $S^\nu$ exists, and in the case

$$S^\nu = S^\nu$$
Case $m^n, \Delta^n$

This bi-algebra has antipode $S(x)$.
Prop. Given a connected graded bialgebra $H = \bigoplus_{n \in \mathbb{N}} H_n$

Recall $H$ has an antipode $S$.

Then $S^{-1}$ exists.

pf. For the bialg $H$, $m^o, \Delta$.

$(\ast)$ is still a connected grading.

So its antipode exists.

But this antipode is $S^{-1}$ by previous comments. $\Box$
Next goal: Given coalgebra $C$

Define a subcoalgebra of $C$

Aside on tensor products

Given $k$-module $V$

Given $k$-submodule $U \subseteq V$

Incl map:

$$i : U \rightarrow V$$

is injective $k$-mod hom.

Consider the $k$-mod hom:

$$u \in U \rightarrow v \in V$$

$$i \circ i : \mathbb{F} \circ \mathbb{F} \rightarrow \mathbb{F} \circ \mathbb{F}$$

$(\star)$ might not be injective, as the next example shows.
Ex  Given

\( F = \text{a field} \)

\( x = \text{indeterminate} \)

\( k = F[x] \text{ polynomials in } x \)

Obs  \( k \) is comm. ring with 1.

let  \( V = \text{vector space over } F \) with dimension 2

pick a basis \( e, f \in V \)

\( V \) becomes a \( k \)-module with \( x \)-action

\( xe = 0, \quad xf = e \)

let  \( U = \text{subspace of } V \) with basis \( e \)

Obs  \( U \) is \( k \)-submodule of \( V \)

Consider incl map

\( i: U \to V \)

\( e \mapsto e \)
Describe $U \oplus U$, $V \otimes V$, $\Theta = \Theta_k$.

$U \oplus U$ is a vector space over $F$.

$U \oplus U$ has basis $eae$.

Moreover,

$x(eae) = (xe)ae = 0$

$V \otimes V$ is a vector space over $F$.

$V \otimes V$ has a basis $eaf = fae$, $faf$.

Moreover,

$x(eaf) = (xe)af = 0$

$x(faf) = (xf)af = eaf$

Observe $eae = (xe)ae$

$= f\circ (xe)$

$= 0$

Now, let's define $\theta : U \oplus U \to V \otimes V$ sends

$eae \to eae$

$0 \to 0$
Ex. Given a $k$-module $V$.

Given a $k$-submodule $U \subseteq V$

Consider each map

$i : U \rightarrow V$

Assume $\exists k$-module $W$ s.t

$V = U + W$ \hspace{1cm} (15)

"$W$ is $k$-module complement of $U$ in $V$"

Then the $k$-module hom

$i @ i : U \otimes U \rightarrow V \otimes V$

is injective.

pf. The $k$-module iso $V = U \otimes W$

induces $k$-module isomorphisms

$V \otimes V \cong (U \otimes W) \otimes (U \otimes W)$

$= (U \otimes U) \oplus (U \otimes W) \oplus (W \otimes U) \oplus (W \otimes W)$

Result follows.
Given $k$-coalgebra $C$

Define a subcoalgebra of $C$

**Naive definition:** A subcoalgebra of $C$ is a $k$-submodule $D$ of $C$ s.t.

$$\Delta_C(D) \subseteq \iota \circ i(D00)$$

Using this def, let $i:D \to C$ into a $k$-coalgebra $C$ s.t.

$$i : D \to C$$

is a coalgebra morph.

First assume $i$ is injective.

Via $i \circ i$, identify $i(i(000))$ with $D00$

Define

$$\Delta_0 : D \to D00$$

$$\Delta_0 : x \to \Delta_C(x)$$

By adjoint $i : D \to C$ is coalgebra morphism.

Next assume $i$ is not injective.

Now the restriction of $\Delta_C$ to $D$ does not induce a map $D \to D00$. $D$ does not inherit a coalgebra from $C$.
So, naive def of subcoalg often works if $i$ is injective.

Here is our official def of subcoalg:

**Def.** Given a $k$-coalg $C$,

A subcoalgebra of $C$ is a $k$-coalg $D$ together with an injective coalg morphim

$\phi : D \to C$

such that

- If we identify $D$ with a $k$-submodule of $C$
- Then $\phi$ becomes the incl map

The def of subbialgebra, subHopf algebra are similar.
Given $u, v, u', v'$

Given surjective $k$-module hom $\phi : V \to V'$

$\phi \circ \psi : u \otimes v \to u' \otimes v'$

$u \otimes v \to \psi(u) \otimes \phi(v)$

has kernel

$\ker(\phi \circ \psi) = \ker(\psi) \otimes V + u \otimes \ker(\phi)$

Moreover, if $K$ is a field, then (*) still holds if the surjectivity assumption is dropped.
LEM Assume $K$ is a field.

Given a $K$-coalgy $C$ and a $K$-module $U$

Given a $K$-module hom $f: C \to U$

Consider the composition

$$C \to C \otimes C \to C \otimes C \otimes C \to C \otimes U \otimes C$$

$\theta: C \to \Delta \quad \text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id}$

$$C \to \sum C \otimes C \to \sum C \otimes C \otimes C \to \sum C \otimes f(C) \otimes C$$

Then $\ker(\theta)$ is a subcoalgy of $C$

pf Since $K$ is field, we need to show

$$\Delta(J) \leq J \otimes J$$

Show both

$$\Delta(J) \leq J \otimes C$$

$$\Delta(J) \leq C \otimes J$$

The following diagrams commute: