So, \( b \) is aligned whereas

\[
\hat{\Psi}_i(b), \quad \hat{\varepsilon}_i(b) \quad \text{even} \quad 1 \leq i \leq r
\]

\[
\hat{\Psi}_r(b) = \hat{\Psi}_m(b)
\]

\[
\hat{\varepsilon}_r(b) = \hat{\varepsilon}_m(b)
\]

In this case

\[
\Psi_i(b) = \frac{\hat{\Psi}_i(b)}{Z}
\]

\[
\varepsilon_i(b) = \frac{\hat{\varepsilon}_i(b)}{Z}
\]

\[
\Psi_r(b) = \Psi_r(b) = \Psi_m(b)
\]

\[
\varepsilon_r(b) = \varepsilon_r(b) = \varepsilon_m(b)
\]

For the moment, assume \( B \) consists of all the aligned \( b \in A \).

Check if \( B \) satisfies axioms \( A_1, A_2 \).
Al: For $x, \eta \in B$ and $\text{is i.s.r}$

**Assume**

$$x \xrightarrow{\epsilon} \eta$$

**Show**

$$\text{wt}(\eta) - \text{wt}(x) = x$$

$$\psi_i(\eta) - \psi_i(x) = 1$$

$$\varepsilon_i(x) - \varepsilon_i(\eta) = 1$$

**Case $\text{is i.s.r}$**

We have

$$x \xrightarrow{i} j \xrightarrow{\omega} \eta$$

$$\text{So}$$

$$\text{wt}(\eta) - \text{wt}(x) = 2 \times x$$

$$\psi(\text{wt}(\eta)) - \psi(\text{wt}(x)) = 4(x)$$

**So** $i$ holds.

**Also**

$$\psi_i(\eta) - \psi_i(x) = 2$$

$$\psi_i(\eta) - 2 \psi_i(x)$$

So $i$ holds.

Sim $i$ holds.
Case $i=r$

We have

\[
\hat{\omega}(y) - \hat{\omega}(x) = \hat{\xi}(r) + \hat{\xi}(r^*)
\]

So

\[
\hat{\Psi}(\hat{\omega}(y)) \quad \hat{\Psi}(\hat{\omega}(x)) \quad \hat{\Psi}(\hat{\xi}(r^*))
\]

So \( \Psi \) holds

Also

\[
\hat{\phi}_r(y) - \hat{\phi}_r(x) = 1
\]

So \( \Psi \) holds

Sim \( \Psi \) holds
A2

For \( b \in B \) and \( i \in s \rho \), show:

\[
\langle \text{wt}(b), \langle \omega^i \rangle^X \rangle = \Psi_i(b) - \varepsilon_i(b)
\]

This follows from:

\[
\text{wt}(b) = \sum_{i=1}^{r} \left( \Psi_i(b) - \varepsilon_i(b) \right) \bar{w}^i X
\]
Next we check that \( B \cup \phi \) is closed under the virtual operation.

For this consider \( \phi \cdot \) - root string

Case 1: is even

String of even length:

\[
\begin{array}{ccccccc}
\psi & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\phi & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
\end{array}
\]

For \( b \in B \), require \( \phi(b), \phi(b) \) even.

Circle the aligned nodes:

\[
\begin{array}{ccccccc}
\psi & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\phi & 3 & 2 & 1 & 0 & & & \\
\end{array}
\]

Get

\[
\begin{array}{ccccccc}
\psi & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\phi & 3 & 2 & 1 & 0 & & & \\
\end{array}
\]
String of odd length:

\[ 0 \ 1 \ 2 \ 3 \ 4 \ 5 \]

\[ 5 \ 4 \ 3 \ 2 \ 1 \ 0 \]

\( \Phi_i \) and \( \Phi_j \) never both even.
None of these nodes aligned.

Conclude \( BV \) is closed under \( \Phi_i \), \( \Phi_j \).

Case \( i = 0 \)

Consider edges in \( B \).

Each connected component is a rectangle.
For a node \( b \) in above rectangle, describe

\[
\hat{\varphi}_r (b), \quad \hat{c}_r (b), \quad \hat{u}_{\text{in}} (b), \quad \hat{m}_r (b)
\]

Since \( \hat{B} \) is seminormal,

\[
\hat{\varphi}_r (b) = W, \quad \hat{\varepsilon}_r (b) = E
\]

\[
\hat{c}_r (b) = S, \quad \hat{\varepsilon}_m (b) = N
\]

For \( b \in B \) require

\[
\hat{\varphi}_r (b) = \hat{\varphi}_{\text{in}} (b), \quad \hat{c}_r (b) = \hat{c}_m (b)
\]

No nodes in rectangle are aligned unless \( B \) is a square, in which case the aligned nodes are circled below:
For each circled node $b_i$,

$\rho_r (a) = \hat{\rho}_r (a) = \hat{\rho}_{mn} (a)$

$\omega_r (a) = \hat{\omega}_r (a) = \hat{\omega}_{mn} (a)$

But $\phi$ is closed under $\rho_r, \omega_r$
We have shown that $BV \delta$ is closed under virtual ops, and is hence a crystal for the $X$ system.

By this, the crystal $B$ is semi-normal.

**Note** As we construct $B$, we do not require that $B$
contain all the aligned elements.

We only require that $BV \delta$ is closed under the virtual ops, and resulting crystal $B$ is SN.

**Def** A **virtual crystal** (in the $X$ system) is
a nonempty subset $B \subseteq \mathcal{V}$ such

$V1$: $\mathcal{V}$ is Skewbridge

$V2$: each $b \in B$ is aligned

$V3$: $BV \delta$ is closed under virtual ops, and $V6 \in B$

$$\psi_i(b) = \max \left\{ k \mid f^*_i(b + \delta) \right\}, \quad \varepsilon_i(b) = \max \left\{ k \mid e_i \delta(b) + b \right\}$$
$E_x \quad r=2 \quad B_2 \text{ vs } D_3$

Recall standard crystal for $D_3$:

\[ \begin{array}{ccc}
  & 3 & 1 \\
1 & \rightarrow & 2 \\
-1 & \rightarrow & 3 \\
\end{array} \]

highest wt is $e_i = \omega_i$

Call this crystal $B_{\omega_i}$

Take $\Lambda_B = B_{\omega_i} \otimes B_{\omega_i}$

For $B_3$, describe the aligned elements and virtual ops.
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</tbody>
</table>
Find \( \hat{\psi}_i, \hat{\xi}_i \) s.t. \( \hat{\beta} \) > \( \bigcirc \) 

\[ \text{\( \bigcirc \) } \geq \text{\( \bigcirc \)} \]

\[ \frac{\text{\( \bigcirc \)}}{\hat{\psi}_i} \quad \frac{\text{\( \bigcirc \)}}{\hat{\xi}_i} \]
Find $\psi_3, \phi_3 \neq \beta$.
Next describe the virtual operators on the aligned elements.