Find \( \hat{\varphi}_2(b \circ c) \), \( \hat{\varphi}_2(b \circ c) \)

(View I) Using signature rule

\[
b \circ c
\]

\[
\begin{array}{ll}
\varphi_2(b) & \varepsilon_2(b) \\
\varphi_2(c) & \varepsilon_2(c)
\end{array}
\]

Cancel (.) to get

\[
\begin{array}{ll}
\varphi_2(b \circ c) & \varepsilon_2(b \circ c)
\end{array}
\]

(View II) Repeating,

\[
b \circ c
\]

\[
\begin{array}{ll}
\varphi_2(b) & \varepsilon_2(b) \\
\varphi_2(c) & \varepsilon_2(c)
\end{array}
\]

\[
\begin{array}{ll}
\psi_1(\psi_1(b)) & \psi_1(\psi_1(b)) \\
\psi_1(\psi_1(c)) & \psi_1(\psi_1(c))
\end{array}
\]

Cancel (.) "in groups of \( \psi_1 \)" to get

\[
\begin{array}{ll}
\psi_1(\psi_1(b \circ c)) & \psi_1(\psi_1(b \circ c))
\end{array}
\]
But $\xi$, $\zeta$ are same so

\[ Y_{\xi} (b \circ c) = Y_{\xi} (b \circ c), \]

\[ \hat{Y_{\xi}} (b \circ c) = Y_{\xi} \hat{E_{\xi}} (b \circ c). \]

V3: $B_1, C$ are crystals

So $B \circ C$ is crystal

Correct function written

\[ E_{\xi}, \quad F_{\xi}, \quad \text{ie } I^x \]

Crystals $B_1, C$ are $SN$ so

$B \circ C$ is $SN$

Also have virtual operators for crystal $B \circ C$:

\[ E_{\iota}, \quad F_{\iota}, \quad \text{ie } I^x \]

Show that in $B \circ C$

\[ e_{\iota} = E_{\iota}, \quad f_{\iota} = F_{\iota}, \quad \text{ie } I^x \]
Either

\[ |\sigma(i)| = 1 \quad \text{or} \quad \delta_i = 1 \]

Assume \( |\sigma(i)| = 1 \)

Write \( \sigma(i) = \{g\} \)

So \( f_i = \frac{g}{g} \)

Find \( F_i (b \otimes c) \)

Sign rule

\[
\begin{array}{c}
\ L \\
\ v_1 \otimes c \\
\vdots \\
v_i(h) \otimes e_i(h) \\
\vdots \\
v_i(c) \otimes e_i(c) \\
\text{cancel } c_1 \text{ } \text{ } i \text{ yet} \\
\vdots \\
\end{array}
\]

\[ \uparrow \]

\[ \ast \]
Case I

\[ \star \text{ is } \]

\[ b \otimes c \]

\[ \Rightarrow c \cdots c \]

\[ \uparrow \]

\[ F_{i}(b \otimes c) = F_{i}(b) \otimes c \]

\[ = b^{\gamma_{c}}(b) \otimes c \]

Case II

\[ \star \text{ is } \]

\[ b \otimes c \]

\[ \Rightarrow c \cdots c \]

\[ \uparrow \]

\[ F_{i}(b \otimes c) = b \otimes F_{i}(c) \]

\[ = b \otimes b^{\gamma_{c}}(c) \]
Find \( f_c(b \circ c) \)

Apply \( f_2^{n_i} \) to \( b \circ c \) \( n_i \) times

Sign rule

\[ b \circ c \]

\[ \cdots \circ \cdots \circ \]

\[ f_3^{(b)} f_3^{(c)} f_3^{(c)} \]

\[ f_3^{(c)} f_3^{(c)} f_3^{(c)} \]

\[ f_3^{(c)} f_3^{(c)} f_3^{(c)} \]

Cancel \((\cdot)\) to get

\[ \cdots \circ \cdots \circ \]

Either

Case I

\[ b \circ c \]

\[ \cdots \circ \cdots \circ \]

\[ f_c(b \circ c) = f_2^{n_i}(b) \circ c \]
Case II

\[ b \otimes c \]

\[ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \]

\[ \vdots \]

\[ f_i(\otimes c) = b \otimes f_i(c) \]

In each case

\[ f \hat{=} F_i \]

For \( i = 1 \) we have

\[ f_i = \prod_{j \in \sigma(1)} f_j \]

and the argument is similar. (\ref{eq})
LEM

Given virtual crystals

\[ B \leq \hat{B} \]
\[ C \leq \hat{C} \]
\[ X \leq Y \]

with \( B, C \) connected.

\[ \begin{bmatrix} \lambda \end{bmatrix} \]

So \( B, C \) have unique \( \lambda \).

Assume

\[ \text{hw } y \mu(B) = \text{hw } y \mu(C) \]

Then crystals \( B, C \) are iso.

If \( B \leq \text{connected comp of } \hat{B} \)
\[ C \leq - \]

WLOG \( \hat{B}, \hat{C} \) connected.

\[ \text{hw element } u \text{ of } B \text{ is hw in } \hat{B} \]

\[ \text{B is obtained by applying virtual ops of } \hat{B} \text{ to } u \]

\[ \text{hw element } v \text{ of } C \text{ is hw in } \hat{C} \]

\[ \text{C is obtained by applying virtual ops of } \hat{C} \text{ to } v \]
\[\text{h.w.} \quad \mathcal{B} = \mathcal{W} \left( \mathcal{U} \right) = \mathcal{V} \left( \mathcal{A} \right) = \mathcal{W} \left( \mathcal{U} \right) = \mathcal{W} \left( \mathcal{V} \right) = \mathcal{C}\]

- crystals \( \mathcal{B}, \mathcal{C} \) are isomorphic

\[\exists \text{ isomorphism} \quad \gamma : \mathcal{B} \to \mathcal{C}\]

By construction \( \gamma \left( \mathcal{U} \right) = \mathcal{U}\)

WLOG identify \( \mathcal{B}, \mathcal{C} \) via \( \gamma \)

so \( \mathcal{U} = \mathcal{V}\)

For \( \gamma \in \mathcal{K}\),

\( \mathcal{B}, \mathcal{C} \) give same virtual reps \( \mathcal{e}_i \)

and same virtual \( \mathcal{f}_i\)

Now by \( \mathcal{K} \times \mathcal{K}\)

\[\mathcal{B} = \mathcal{C}\]

result follows.
Motivation

For $\overline{\Lambda} = \Lambda_r$, $GL(m)$ and for each partition $\lambda \in \Lambda^+$ we defined a crystal $B_\lambda$. $B_\lambda$ is seminormal, connected, has unique h-vecm, $h \lambda$

Next, consider any other root system $\Phi$ from the classification: $B_r, C_r, D_r, E_6, E_7, E_8, F_r, G_2$

Take $\Lambda = \Lambda_{\text{root}}$

For $\lambda \in \Lambda^+$ we wish to define a certain connected seminormal crystal $B_\lambda$ with a unique h-vecm and $h \lambda$

To do this, we first define $B_\lambda$, assuming $\lambda = \overline{\lambda}$ is a fundamental dominant weight. Such a $B_\lambda$ is called a fundamental crystal.
For $E = B$, $C_r$, $D_r$ consider the standard crystal $B = B^\infty$.

For $1 \leq k \leq r$ consider

$$u = k \otimes (k+1) \otimes \ldots \otimes z \otimes 1 \in B^\otimes k$$

Note

$$w(u) = e_i + e_r - e_r$$

**LEM** With above notation,

(i) $u$ is an even in the crystal $B^\otimes k$

(ii) For $E = B_r$,

$$w(u) = \begin{cases} \overline{w_k} & \text{if } 1 \leq k < r \\ \overline{2w_r} & \text{if } k = r \end{cases}$$

(iii) For $E = C_r$,

$$w(u) = w_k \quad \text{is even}$$

(iv) For $E = D_r$,

$$w(u) = \begin{cases} \overline{w_k} & \text{if } 1 \leq k < r - 2 \\ \overline{w_{k+2}} + \overline{w_r} & \text{if } k = r - 2 \\ \overline{2w_r} & \text{if } k = r \end{cases}$$

**pf** (i) Roughly check $\varepsilon_i(u) = 0$ for $1 \leq i < r$ using some rule.

(iii)-(iv) Use formula $f_{\overline{w_k}}$
LEM  With above notation, let

\[ C = \text{the connected component of the crystal } B^{\alpha k} \]

that contains \( u \).

Then

(i) For \( \tilde{E} = B_r \), \( C \) is a virtual crystal

(ii) For \( \tilde{E} = B_r \), \( C \) is Steenrod

(iii) \( u \) is unique hom vector in \( C \)

pf (i) \( B \) is virtual so \( B^{\alpha k} \) is virtual

(ii) \( B \) is Steenrod so \( B^{\alpha k} \) is Steenrod

(iii) By (i), (ii) \( \square \)
For $\bar{F} = D_r$

Consider

\[ v = \begin{array}{c}
\bar{r} \\
\bar{s} \\
\bar{t} \\
\bar{u} \\
\bar{v}
\end{array} \in B^\oplus r \]

Note

\[ wt(v) = e_1 + e_r + \cdots + e_r - e_r \]

LEM With above notation

(i) $v$ is hw vector in crystal $B^\oplus r$

(ii) $wt(v) = 2 wt(v)$

PF Rovee
LEM With above notation, let

\[ C = \text{the connected component of the crystal } B^r \text{ that contains } \nu. \]

Then

(i) \( C \) is Steinbrucke

(ii) \( \nu \) is unique hw vector of \( C \).

pf (i) \( B \) is Steinbrucke so \( B^r \) is Steinbrucke

(ii) By (i)