Chapter II  
TD pairs and Leonard pairs

For this chapter the field $\mathbb{F}$ is arbitrary.

Fix integer $d \geq 0$

$\text{Mat}_d(\mathbb{F})$ is the $\mathbb{F}$-algebra of all $d \times d$ matrices with entries in $\mathbb{F}$. Index the rows/columns by $0, 1, \ldots, d$.

Given a bidiagonal matrix $A \in \text{Mat}_d(\mathbb{F})$;

\[
A = \begin{pmatrix}
    a_0 & b_0 & 0 & 0 \\
    c_0 & a_1 & b_1 & 0 \\
    & c_1 & \ddots & \ddots \\
    & & & c_{d-1} & a_d \\
    & & & & c_d
\end{pmatrix}
\]

Call $A$ irreducible whenever $c_i b_i \neq 0$ for $i = 0, 1, \ldots, d$.

Until further notice assume $A$ is irreducible.
Note that \( 0 \leq i, r \leq d \)

\[
(A^r)^*_{ij} = \begin{cases} 
0 & \text{if } |i - j| > r \\
\ne 0 & \text{if } |i - j| = r
\end{cases}
\]

Therefore

\[ \{ A^r \}_{r=0}^d \text{ are linearly independent} \]

Therefore

the min poly of \( A \) = the char poly of \( A \)

Therefore

each eigenspace of \( A \) has dim \( 1 \)

(\text{Caution: possibly A is not diagonalizable})

For \( 0 \leq i \leq d \) define \( E_i^x \in \mathbb{M}_{d \times d} \) by

\[ E_i^x = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0) \]

\( \uparrow \text{coor } i \)

So

\[ E_i^x E_j^x = \delta_{ij} E_i^x \quad (0 \leq i, j \leq d) \]

\[ I = \sum_{i=0}^d E_i^x \]

Note that \( 0 \leq i, r \leq d \)

\[ E_i^x A^r E_j^x = \begin{cases} 
0 & \text{if } |i - j| > r \\
\ne 0 & \text{if } |i - j| = r
\end{cases} \]

In particular \( 0 \leq i, d \)

\[ E_i^x A E_j^x = \begin{cases} 
0 & \text{if } |i - j| > 1 \\
\ne 0 & \text{if } |i - j| = 1
\end{cases} \]
LEM 1  The following is a basis for the $\mathbb{F}$-vector space $\text{Mat}_{d \times d}(\mathbb{F})$:

$$A^{\otimes d} E^x o \in \mathbb{F} d$$

If $A^{\otimes d} E^x o$ is the $(i,j)$-entry,

$$(A^{\otimes d} E^x o)_{ij} = (A^i)_o (A^j)_o$$

$$= \begin{cases} 
0 & \text{if } i > r \text{ or } j > a \\
+1 & \text{if } i = r \text{ and } j = a 
\end{cases}$$

Result follows.  \( \square \)

COR 2  The $\mathbb{F}$-algebra $\text{Mat}_{d \times d}(\mathbb{F})$ is generated by $A^{\otimes d} E^x o$.

pf  By LEM 1  \( \square \)
Define

\[ k_i = \frac{b_0 b_1 \ldots b_{i-1}}{c_1 \cdot c_2 \ldots \cdot c_i} \quad 0 \leq i \leq d \]

So

\[ k_0 = 1 \]

\[ k_i \rightarrow 0 \quad 0 \leq i \leq d \]

Define \( k \in \text{Mat}_{d \times d}(\mathbb{F}) \) by

\[ k = \text{diag}(k_0, k_1, \ldots, k_d) \]

LEM 3: We have

\[ A^t k = k A \]

pf routine \( \square \)
For any $F$-algebra $A$, by an anti automorphism of $A$ we mean an $F$-linear bijection $\sigma: A \to A$ that reads

$$(ab)^\sigma = b^\sigma a^\sigma \quad \forall a, b \in A$$

**LEMMA.** There is a unique anti automorphism $\dagger$ of $\text{Mat}_n(F)$ that fixes $A$ and each of $E_0^*, E_1^*, \ldots, E_n^*$. Moreover $\dagger^2 = 1$. 

**Proof.** Existence: The map

$$\dagger: \text{Mat}_n(F) \to \text{Mat}_n(F)$$

$$\dagger: B \mapsto K^*B^*K$$

is an anti automorphism of $\text{Mat}_n(F)$ that meets the requirements.

Note that $\dagger^2 = 1$.

**Uniqueness:** Let $\sigma$ denote any anti automorphism of $\text{Mat}_n(F)$ that meets the requirements.

The composition $\sigma \dagger$ is an anti automorphism of $\text{Mat}_n(F)$ that fixes $A, E_0^*, \ldots, E_n^*$. Now $\sigma \dagger = 1$ by LEM 2. So $\sigma = \dagger$. \[\Box\]
Consider the $F$-vector space
\[ V = F^{d \times 1} \] (column vectors)

Index the rows $0, 1, \ldots, d$

$V$ is a module for $\text{Mat}_{d \times d}(F)$

**LEM 5** (i) There is a non-zero bilinear form
\[ \langle \cdot , \cdot \rangle : V \times V \to F \]

such that
\[ \langle Bu, v \rangle = \langle u, B^tv \rangle \]

for every $u \in V$ and $B \in \text{Mat}_{d \times d}(F)$

(ii) $\langle \cdot , \cdot \rangle$ is symmetric and non-degenerate.

(iii) $\langle \cdot , \cdot \rangle$ is unique up to mart by a non-zero scalar in $F$

**pf (i):** Define
\[ \langle \cdot , \cdot \rangle : V \times V \to F \]
\[ u \quad v \quad \mapsto u^t K v \]

By LEM 4
\[ \langle Bu, v \rangle = u^t B^t K v = u^t K v^t B^t K v = u^t K \beta^t v = \langle u, \beta^tv \rangle \]

(iii) clear

(iii) Routine.
Note that if \( u, v \in V \)

\[
\langle Au, v \rangle = \langle u, Av \rangle
\]

\[
\langle E_i^x u, v \rangle = \langle u, E_i^x v \rangle \quad 0 \leq i \leq d
\]

Let \( A^x \) denote a diagonal matrix in \( \text{Mat}_{d \times d}(\mathbb{F}) \).

Write

\[
A^x = \text{diag}(\theta_0^x, \theta_1^x, \ldots, \theta_d^x)
\]

So

\[
A^x = \sum_{i=0}^{d} \theta_i^x E_i^x
\]

Observe \( A^x^T = A^x \)

\[
\langle A^x u, v \rangle = \langle u, A^x^T v \rangle \quad \forall u, v \in V
\]

**LEM 6** Assume \( \theta_0^x \neq \theta_i^x \) for all \( i \neq 0 \).

(i) \( E_0^x = \prod_{i=0}^{d} \frac{A^x - \theta_i^x I}{\theta_0^x - \theta_i^x} \)

(ii) \( A, A^x \) generate \( \text{Mat}_{d \times d}(\mathbb{F}) \)

(iii) There exist \( W \subseteq V \) s.t.

\[
W \neq 0, \quad W \perp V, \quad A W \subseteq W, \quad A^x W \subseteq W
\]

pf Routine
Comments on TD pairs

Until further notice

\[ V = \text{vector space over } \mathbb{F} \text{ with finite positive dimension} \]

Let \( A, A^* \) denote a TD pair on \( V \)

**DEF 7** Let \( V' \) denote a vector space over \( \mathbb{F} \) with finite pos dimension.

Let \( A', A'^* \) denote a TD pair on \( V' \).

By an isomorphism of TD pairs from \( A, A^* \) to \( A', A'^* \) we mean an \( \mathbb{F} \)-linear bijection \( \sigma : V \to V' \) that satisfies

\[ A \sigma = \sigma A, \quad A^* \sigma = \sigma A^* \]

---
An ordering \( \{ V_i \}_{i=0}^{d} \) of the eigenspace \( \lambda A \)

is called **standard** whenever

\[
A^* V_i \leq V_i + V_i + V_{i+1} \quad 0 \leq i \leq d
\]

where \( V_0 = 0, \quad V_d = 0 \).

If \( \{ V_i \}_{i=0}^{d} \) is standard then so is \( \{ V_{d-i} \}_{i=0}^{d} \)

and no further ordering is standard.

Similar concepts apply to \( A^* \)
DEF 8. A \textit{triangular system} on \( V \) is a sequence
\[
\mathcal{E} = \left( A; \{ E_i \}_{i=0}^d; A^*; \{ E'_i \}_{i=0}^d \right)
\]
such that

(i) \( A, A^* \) is a TD pair on \( V \)

(ii) \( \{ E_i \}_{i=0}^d \) is a standard ordering of the primitive idempotents of \( A \)

(iii) \( \{ E'_i \}_{i=0}^d \) is a standard ordering of the primitive idempotents of \( A^* \)

We say \( \mathcal{E} \) is a \( \mathcal{T} \). We call \( V \) the underlying vector space. We call \( A, A^* \) the associated TD pair.
We mention some special cases of TD systems.

Let $\Phi$ denote a TD system on $V$, as in DEF 8.

One checks the following are equivalents:

(i) $d = 0$
(ii) $\delta = 0$
(iii) $A \in I F I$
(iv) $A^* \in I F I$
(v) $\dim V = 1$

$\Phi$ is called trivial whenever (i)-(v) hold.

A Leonard system is a TD system such that:

- $\dim E_i V = 1$, $0 \leq i \leq d$
- $\dim E_i^* V = 1$, $0 \leq i \leq S$

In this case, $d = S = \dim V - 1$. 

Given a TD system \( E \) on \( V \), as in Def 8.

Then each of the following is a TD system on \( V \):

\[
\overline{E}^V := \left( A; E_i \right)_i \circ \left( A^*; \{ E_{i,j} \}_{i,j=0}^k \right),
\]

\[
\overline{E}^\Psi := \left( A; \{ E_{i,j} \}_{i,j=0}^k \right) \circ \left( A^*; \{ E_i \}_{i=0}^k \right),
\]

\[
\overline{E}^\Psi^* := \left( A^*; \{ E_i \}_{i=0}^k \right) \circ \left( A; \{ E_{i,j} \}_{i,j=0}^k \right).
\]

**Obs**: 

\[
\downarrow \downarrow = \Psi \downarrow, \quad \downarrow \Psi = \Psi \downarrow, \quad \Psi \Psi = \Psi \downarrow
\]

\[
\star^2 = 1, \quad \downarrow^2 = 1, \quad \Psi^2 = 1
\]

The group gen by symbols \( \downarrow, \Psi, \star \) subject to the above relations is the Dihedral group \( D_4 \).

\( D_4 \) has 8 elements. \( D_4 \) is the group of symmetries of a square.

So \( D_4 \) acts on the set of all TD systems.

TD systems in the same \( D_4 \)-orbit are called **relatives**.
Until further notice for a TD system in $V$:

$$
\Phi = (A; \{E_i; i=0\}; A^*; \{E_i^*; i=0\})
$$

**Obs**

$$
E_i^* A E_i = 0 \quad \text{if} \quad |i-j| > 1 \quad (0 \leq i, j \leq d) \\
E_i^* A E_i = 0 \quad \text{if} \quad |i-j| > 1 \quad (0 \leq i \leq d)
$$

**DEF** 9. For $0 \leq i \leq d$ let $\theta_i$ denote the eigenvalue of $A \Phi E_i$.

For $0 \leq i \leq d$ let $\theta_i^*$ denote the eigenvalue of $A^* \Phi E_i^*$.

Call $\{\theta_i; i=0\}$ the **eigenvalue sequence** of $\Phi$.

Call $\{\theta_i^*; i=0\}$ the **dual eigenvalue sequence** of $\Phi$.

**Obs**

$$
A = \sum_{i=0}^{d} \theta_i E_i, \quad A^* = \sum_{i=0}^{d} \theta_i^* E_i^*
$$

By construction, $\{\theta_i; i=0\}$ are mutually distinct scalars in $IF$.

$\{\theta_i^*; i=0\}$
DEF 10. Call $E$ bipartite whenever

$$E_i^* A E_i^* = 0 \quad (0 \leq i \leq 5)$$

Call $E$ dual bipartite whenever $E^*$ is bipartite.

i.e.

$$E_i A^* E_i^* = 0 \quad (0 \leq i \leq 8)$$

Call $E$ totally bipartite whenever $E$ is bipartite and dual bipartite.

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EXAMPLE II. For $E$ trivial, $A = 0$.

- $E$ is bipartite if $A = 0$.
- $E$ is dual bipartite if $A^* = 0$.
- $E$ is totally bipartite if $A = A^* = 0$.

Next goal: Classify the totally bipartite TO systems up to isomorphism.

We use a conceptual approach with minimal computation.