We continue to discuss our totally bipartite TD system \( \mathcal{E} = (A, \{E_i \mid i = 0\}; A^*, \{E_i^* \mid i = 0\}) \) on \( V \).

We saw earlier that \( \mathcal{E} \) is a Leonard system.

**LEM 23**  For \( \beta, \rho \in \mathbb{F} \) TFAE:

1. \( \theta_i^2 - \beta \theta_i + \theta_i = \rho \) \( 1 \leq i \leq d \)

2. \( A^2 - \beta AA^* + A^*A^2 = \rho A^* \)

**Proof** Define \( C = A^2 - \beta AA^* + A^*A^2 - \rho A^* \)

View \( C = \sum_{i=0}^{d} E_i C E_i \)

So \( C = 0 \iff E_i C E_i = 0 \) for all \( i \in \{0, 1, \ldots, d\} \).

For \( 0 \leq i \leq d \)

\[ E_i C E_i = E_i A^* C E_i \left( \theta_i^2 - \beta \theta_i \theta_i + \theta_i^2 - \rho \right) \]

Also \( E_i A^* E_i = 0 \) \( \iff \mid i - 21 = 1 \)

Result follows. \( \square \)
Define $M = \text{subalgebra of } \text{End}(V) \text{ gen by } A$
\[ M^* = \ldots A^* \]

Define
\[ MA^*M = \text{Span} \{ uA^*v \mid u, v \in M \} \]

**LEM 2.4** The following is a basis for the $\mathbb{F}$-vector space $MA^*M$:
\[ E_i A^* E_j \quad |i-j| = 1 \quad (0 \leq i, j \leq d) \quad (*) \]

Moreover, the dimension of $MA^*M$ is $2d$

Proof: the elements $(*)$ span $MA^*M$ since $E_0, \ldots, E_d$
span $M$ and $E_i A^* E_j = 0$ if $|i-j| \neq 1$.

Using $E_i E_j = 0$ for $i, j \neq d$ we check that $(*)$ are linearly independent. So $(*)$ is a basis for $(*)$.

This basis has $2d$ elements. \[ \square \]
LEM 25

(i) \[ E_0 A^* = E_0 A^* E_i \]

(ii) \[ E_i A^* = E_i A^* E_{i-1} + E_i A^* E_{i+1} \quad (i \leq i \leq d) \]

(iii) \[ E_d A^* = E_d A^* E_{d-1} \]

pf

Fa 0\leq i \leq d iew

\[ E_i A^* = E_i A^* I \quad I = \sum_{j=0}^{d} E_j \]

Recall

\[ E_i A^* E_i = 0 \quad \forall \ 1 \leq i \leq d \]

\[ \square \]
LEM 26

(i) \[ A^{*}E_0 = E_r A^{*}E_0 \]

(ii) \[ A^{*}E_r = E_{ir} A^{*}E_r + E_{i r} A^{*}E_r \quad (i \neq r \neq d, r) \]

(iii) \[ A^{*}E_d = E_{d r} A^{*}E_d \]

pf Apply the anti-automorphism \( \dagger \) to everything in LEM 25.
Lemma 27

For $1 \leq i \leq d$,


Moreover

\[ \sum_{0 \leq i \leq d \text{ even}} E_i A^* = \sum_{0 \leq i \leq d \text{ odd}} A^* E_i \]

\[ \sum_{0 \leq i \leq d \text{ odd}} E_i A^* = \sum_{0 \leq i \leq d \text{ even}} A^* E_i \]

Proof: Solve the equations in Lemma 25, 26. \( \Box \)
LEM 28  

The following is a basis for the $F$-vector space $MA^*M$:

\[ E, A^*, A^*E_i, \quad 0 \leq i \leq d - 1 \]

\( (\star) \)

pf

By construction

\[ \text{Span}(\star) \subseteq MA^*M \]

By LEM 24 and LEM 27

\[ MA^*M \subseteq \text{Span}(\star) \]

So

\[ MA^*M = \text{Span}(\star) \]

There are $2d$ elements in $\star$. These elements are linearly independent since $MA^*M$ has dimension $2d$. 

\[ \square \]
COR 29

\[ MA^*M = MA^*A^*M \]

pf \leq \text{ By LEM 28}

\geq \text{ clean}

10/11/13
Define

\[(MA^*M)^{\text{sym}} = \left\{ u \in MA^*M \mid u^+ = u \right\}\]

"symmetric part of $MA^*M$"

**LEM 30** The following is a basis for the

$F$-vector space $(MA^*M)^{\text{sym}}$:

\[E_i A^*E_i \pm E_i A^*E_i, \quad 1 \leq i \leq d\]

Moreover, the dimension of $(MA^*M)^{\text{sym}}$ is $d$.

**pf** By LEM 24 and since

\[(E_i A^*E_i)^+ = E_i A^*E_i \quad \text{for} \quad 1 \leq i \leq d\]

$\blacksquare$
LEM 31 The following is a basis for the $F$-vector space $(MA^*M)^{sym}$:

$$E_i A^* + A^* E_i$$

$0 \leq i \leq d$

pf By Lem 28 and some

$$(E_i A^*)^+ = A^* E_i$$

$$(A^* E_i)^+ = E_i A^*$$

$0 \leq i \leq d$
LEM 32

\[(MA^xM)^{\text{sym}} = \left\{ uA^x + A^x u \mid u \in M \right\} \]

pf \:
\begin{align*}
\leq & \:	ext{By LEM 31} \\
\geq & \:orall u \in M \\
(A^x u)^+ & = u A^x \\
(u A^x)^+ & = A^x u \\
\tag{So \: it \: leaves} \\
& u A^x + A^x u \\
\text{invariant.} \\
\end{align*}

\[\square\]
LEM 33  The $F$-vector space

$$(M^*M)^{sym}$$ is spanned by


Pf  
M has basis $\{A^i\}_{i=0}^d$ as by LEM 32

$$(M^*M)^{sym} = \text{Span}\{ A^iA^*+A^*A^i \}_{i=0}^d$$

$$\leq \text{Span}(\#)$$

Also

$$\text{Span}(\#) \leq (M^*M)^{sym}$$

by LEM 32 and the def of $(M^*M)^{sym}$.  \(\square\)

---

There are all vectors in $(\#)$ and

$(M^*M)^{sym}$ has dimension $d$.

So the vectors $(\#)$ are linearly dependent.

We now find this dependency.
LEM 34

(i) \( \text{char } F \neq 2 \)

(ii) \( \exists \) integer \( n \geq 1 \) such that \( \theta_n^* = -\theta_0^* \)

(iii) \( A^*A^* + A^*A^* \) is in the span of
\[ A^*, \AA^*A^*, \AA^*A^*A^*, \ldots, A^*A^* + A^*A^*A^* \]

pf In LEM 33 the vectors \((*)\) are linearly dependent.

So \( \exists \) scalar \( \{ \alpha_i \}_{i=0}^d \) in \( F \), not all 0, such that
\[ \lambda_0 A^* + \sum_{i=1}^d \alpha_i \left( A^*A^* + A^*A^*A^* \right) = 0 \]

\( A^* \neq 0 \) since \( \overrightarrow{x} \) is non-trivial.

So \( \exists i^* \) \((1 \leq i^* \leq d)\) such that \( \alpha_i^* \neq 0 \). Define
\[ n = \max \left\{ i^* \mid 1 \leq i^* \leq d, \alpha_i^* \neq 0 \right\} \]

So \( \lambda_0 A^* + \sum_{i=1}^d \alpha_i \left( A^*A^* + A^*A^*A^* \right) = 0 \), \( \alpha_n \neq 0 \)

Now
\[
0 = E_0^* \left( \lambda_0 A^* + \sum_{i=1}^d \alpha_i \left( A^*A^* + A^*A^*A^* \right) \right) E_0^*
\]
\[
= \left[ E_0^* A^* E_0^* \right] \left[ \theta_n^* + \theta_0^* \right] = 0 \text{ if } i^* < n
\]

So \( \theta_n^* + \theta_0^* = 0 \)

So \( \theta_n^* = -\theta_0^* \)

Now when \( F \neq 2 \) else \( \theta_n^* = \theta_0^* \) end. Result follows.

\( \square \)
Referring to LEM 34, we will show \( n = d \).

Prop 35 \( \exists \) scalars \( \beta, p, p^* \) in \( \mathbb{F} \) such that

\[
A^* A - \beta A A^* + A^* A^2 = \rho A^*
\]

(1)

\[
A^* A - \beta A^* A A^* + A A^* = p^* A
\]

(2)

The sequence \( \beta, p, p^* \) is unique if \( d \geq 3 \).

\[ p \]

\( \text{Observe} \)

\[
A A^* A \in (M A^* M)^{\text{sym}}
\]

By LEM 33 and LEM 34 \( \exists \{\delta_i\}_{i=0}^d \) in \( \mathbb{F} \)

with \( \delta_0 = 0 \) such that

\[
A A^* A = \delta_0 A^* + \sum_{i=1}^d \delta_i \left( A^i A^* + A^* A^i \right)
\]
Claim 1: \( \alpha_i = 0 \quad 3 \leq i \leq d \)

Proof: Suppose not, then \( d \geq 3 \).

Define \( t = \max \{ i \mid 3 \leq i \leq d, \alpha_i \neq 0 \} \).

Then \( t \neq n \) and

\[
AA^*A = \alpha_0 A^* + \sum_{i=1}^{t} \alpha_i \left( A_i^* A_i + A_i A_i^* \right)
\]

So,

\[
0 = E_0^* \left( \alpha_0 A^* + \sum_{i=1}^{t} \alpha_i \left( A_i^* A_i + A_i A_i^* \right) - A A^* A \right) E_0
\]

\[
= \underbrace{E_0^* A t E_t^*}_{\#} \# t \left( e_t^* + \theta_0^* \right) \quad \text{by L20}
\]

\[
\# 0 \quad 0 \quad 0 \text{ by } t \neq n
\]

\[\neq 0 \]

End, claim proved.
Claim 2: \( \exists \beta \in \mathbb{F} \) such that

\[ \theta_i^\times - \beta \theta_i^\times + \sigma_i^\times = 0 \quad \text{for } i \leq d. \]

pf: Assume \( d \geq 2 \); otherwise there is nothing to show.

By claim 1,

\[ 0 = x_0^A + x_1(A^A)^{-1} A^A x_0^A + x_2(A^A)^{-1} A^A x_2^A - x_3 A^A x_4. \]

So in \( 1 \leq i \leq d \),

\[ 0 = E_i^\times (x_0^A + x_1(A^A)^{-1} A^A x_0^A) + x_2(A^A)^{-1} A^A x_2^A - x_3 A^A x_4. \]

\[ \text{[Use LEM 20]} \]

\[ = E_i^\times A^2 E_i^\times (x_0^A + x_2(A^A)^{-1} A^A x_2^A - x_3 A^A x_4). \]

\[ \text{[Use LEM 20]} \]

So

\[ x_2(\theta_i^\times + \sigma_i^\times) = \theta_i^\times \quad \text{for } 1 \leq i \leq d. \]

Assume \( x_2 \neq 0 \). Put \( \beta = \frac{1}{x_2} \) and we are done.

Assume \( x_2 = 0 \). Then

\[ \theta_i^\times = 0 \quad \text{for } 1 \leq i \leq d. \]

So \( d = 2 \) and

\[ \theta_i^\times = 0. \]

In no case \( n = 2 \) and

\[ \theta_2^\times = -\theta_0^\times. \]

Let \( \beta \in \mathbb{F} \) be arbitrary. Then

\[ \theta_0^\times - \beta \theta_1^\times + \theta_2^\times = 0. \]

And we are done, claim proved.
By Lem 21 and Lem 21,

\[ \exists \rho \in F \text{ such that } \]

\[ \theta_i^2 - \beta \theta_i \theta_i + \theta_i = \rho \quad 1 \leq i \leq d \]

Now by Lem 23 (applied to \( \bar{\theta}^* \))

\[ A^*A - \beta A^*AA^* + A^*A^2 = \rho A^* \]

Now for \( 1 \leq i \leq d \)

\[ 0 = E_i \left( A^*A - \beta A^*AA^* + A^*A^2 - \rho A^* \right) E_i \]

[applying L20 to \( \bar{\theta}^* \)]

\[ = E_i A^*E_i \left( \theta_i - \beta \theta_i + \theta_i \right) \]

\[ = E_i A^*E_i \left( \theta_i \right) \]

\[ = 0 \]

\[ \theta_i - \beta \theta_i + \theta_i = 0 \quad 1 \leq i \leq d \]

By Lem 21 \( \exists \rho \in F \text{ such that } \)

\[ \theta_i^2 - \beta \theta_i \theta_i + \theta_i = \rho \quad 1 \leq i \leq d \]

Now by Lem 23

\[ A^*A - \beta A^*AA^* + A^*A^2 = \rho A^* \]
Now assume $d \geq 3$.

We show that the sequence $\beta, \rho, \rho^*$ is unique.

By LEM 23

\[ \theta_{i+1}^2 - \beta \theta_i \theta_{i+2} + \theta_{i+2}^2 = \rho \quad \text{for all } i \quad (\star) \]

By LEM 22

\[ \theta_{i-1}^2 - \beta \theta_i \theta_{i+1} = 0 \quad \text{for all } i \quad (\star \star) \]

The scalar $\beta$ is determined by $(\star \star)$.

The scalar $\rho$ is determined by $(\star)$.

The scalar $\rho^*$ is similarly determined. \(\square\)
We record some results from the proof of Prop 35.

LEM 36. Given \( \beta, \rho, \rho^* \in \mathbb{R}^n \) that satisfy Prop 35, then

\[ (i) \quad \theta_i^2 - \beta \theta_i \theta_i^x + \theta_i^2 = \rho \quad 1 \leq i \leq d \]

\[ (i') \quad \theta_i^x - \beta \theta_i^x \theta_i^x + \theta_i^x = \rho^* \quad 1 \leq i \leq d \]

\[ (iii) \quad \theta_i - \beta \theta_i + \theta_i = 0 \quad 1 \leq i \leq d+ \]

\[ (iv) \quad \theta_i^x - \beta \theta_i^x + \theta_i^x = 0 \quad 1 \leq i \leq d+ \]

\[ \square \]