Fall 2013
Math 846

TRIDIAGONAL PAIRS and Related Topics

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Introduction

$F$: a field

$V$: a vector space over $F$ with finite positive dim

$\text{End}(V) = \{ A : V \rightarrow V \mid A \text{ is } F\text{-linear} \}$

For $A \in \text{End}(V)$ and a subspace $W \subseteq V$

$W$ is an eigenspace for $A$ whenever $W \neq \emptyset$

and $\exists \lambda \in F$ such that

$W = \{ v \in V \mid Av = \lambda v \}$

$A$ is diagonalizable whenever $V$ is spanned

by the eigenspaces of $A$
Def A tri-diagonal pair (a T0 pair)
on V is an ordered pair \( A, A^* \) of elements
in \( \text{End}(V) \) such that

(i) Each of \( A, A^* \) is diagonalizable

(ii) \( \exists \) an ordering \( \{ V_i \} \) of the eigenspaces of \( A \)
such that

\[ A V_i = V_{i-1} + V_i + V_{i+1} \quad 0 \leq i \leq d \]

where \( V_0 = V_0^* = 0 \)

(iii) \( \exists \) an ordering \( \{ V_i^* \} \) of the eigenspaces of \( A^* \)
such that

\[ A^* V_i^* = V_{i-1}^* + V_i^* + V_{i+1}^* \quad 0 \leq i \leq s \]

where \( V_s^* = V_s^* = 0 \)

(ii) There does not exist a subspace \( W \subseteq V \) such that

\[ A W = W, \quad A^* W = W, \quad W \neq 0, \quad W \neq V \]

We say \( A, A^* \) is over \( T_0 \). Call \( V \) the underlying vector space
Note: $A^*$ does not mean conjugate-transpose.

$V^*$ does not mean dual space.

$A, A^*$ are arbitrary linear trans. that satisfy (i)-(iv).
Some basic facts about TD pairs

(proved later)

Given TD pair $A, A^*$ on $V$ as in Def 1

It turns out

$d = S$

Call this the **diameter** of the pair
Fact 1

For $0 \leq i \leq d$ let

$$\theta_i = \text{eigenvalue of } A \mapsto V_i,$$
$$\theta_i^* = \text{eigenvalue of } A^* \mapsto V_i^*$$

Then

$$\frac{\theta_{i+2} - \theta_{i+1}}{\theta_i - \theta_i^*}$$
$$\frac{\theta_{i+2} - \theta_i}{\theta_{i+1} - \theta_i^*}$$

are equal and independent of $i$ for $2 \leq i \leq d - 1$

Note

Let $\overline{F} = \text{algebraic closure of } F$

The "most general" solution to above recurrence is

$$\theta_i = a + b q^i + c q^{-i}$$
$$\theta_i^* = a^* + b^* q^i + c^* q^{-i}$$

where

$q, a, b, c, a^*, b^*, c^* \in \overline{F}, \quad q \neq 0, \pm 1$
Fact II

There exist scalars $\beta, \gamma, \delta, \delta^*$ in $F$ such that

\[
\begin{bmatrix}
A, & A^2A^* - \beta AA^*A + A^*A^2 - \gamma(A^*AA^*A) - \delta A^*
\end{bmatrix} = 0,
\]

\[
\begin{bmatrix}
A^*, & A^*A - \beta A^*AA^* + AA^* - \gamma^*(A^*A + AA^*) - \delta A
\end{bmatrix} = 0
\]

where

\[
[xy] = xy - yx
\]

"TD relations"

Note: Assume $\gamma = \gamma^* = 0, \delta = \delta^* = 0, \beta = q^\frac{q+1}{2}$, $0 \neq q \in F, q \neq -1$

TD rules become

\[
A^3A^* - [3]_f A^2A^*A + [3]_q AA^*A^2 - A^*A3 = 0,
\]

\[
A^3A - [3]_f A^*AA^* + [3]_q A^*AA^* - AA^*3 = 0
\]

where \([n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}\)

"cubic $q$-Serre relations"
Fact III

For $0 \leq i \leq d$ define

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d)$$

Then

$$V = \sum_{i=0}^{d} U_i \quad \text{(direct sum)}$$

Also

$$(A - \Theta_i I) U_i \subseteq U_{i-1} \quad \text{for } i \leq d$$

$$(A - \Theta_d I) U_d = 0$$

$$(A^* - \Theta_i^* I) U_i \subseteq U_{i-1} \quad \text{for } i \leq d$$

$$(A^* - \Theta_0^* I) U_0 = 0$$

"Split decomposition"
Fact IV

For $0 \leq i \leq d$

$$\dim V_i = \dim V_i^* = \rho_i$$

Moreover, the sequence $\{\rho_i \}_{i=0}^d$ is symmetric and unimodal, i.e.

$$\rho_i = \rho_{d-i}, \quad 0 \leq i \leq d$$

$$\rho_{i-1} \leq \rho_i, \quad 1 \leq i \leq d$$

Also

$$\rho_i \leq \rho_0 \binom{d}{i}, \quad 0 \leq i \leq d$$

where $\binom{d}{i}$ is the binomial coefficient.

The sequence $\{\rho_i \}_{i=0}^d$ is called the shape $A$.

$A^* \text{ is called sharp whenever } \rho_0 = 1$
FACT V

If $F$ is algebraically closed then $A, A^\times$ is sharp.
FACT II

Assume $A, A^*$ is sharp.

Then, there exists a nondegenerate, symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $V$ such that both

$$\langle Au, v \rangle = \langle u, Av \rangle$$

$$\langle A^*u, v \rangle = \langle u, A^*v \rangle$$

for all $u, v \in V$. 
Fact VII

The Leonard pairs are classified up to isomorphism.

They are in bijection with a family of orthogonal polynomials consisting of:

- $q$-Racah
- $q$-Hahn
- dual $q$-Hahn
- $q$-Krawtchouk
- dual $q$-Krawtchouk
- affine $q$-Krawtchouk
- quantum $q$-Krawtchouk
- Racah
- Hahn
- dual Hahn
- Krawtchouk
- Bannai/Ito

Orphans (char $\mathbb{F} = 2$ only)

This family is the terminating branch of the Askey scheme

of orthogonal polynomials
FACT VIII

the sharp TD pairs are classified up to ISO.
Connections

**Linear algebra:**
- Split decomposition
- Askey-Wilson relations
- Tridiagonal relations

**Combinatorics:**
- $q$-polynomial distance-regular graphs
- Uniform posets

**Double affine Hecke algebra of type $(\mathfrak{sl}_2, \mathfrak{sl}_2)$**

**Lie algebra:**
- $\mathfrak{sl}_2$
- $\mathfrak{sl}_2$-loop algebra
- Onsager algebra $\Theta$
- Tetrahedron algebra $\mathcal{T}$

**Special functions + Orthogonal polynomials:**
- $q$-Racah polynomials + relations in Askey scheme
- (basic) hypergeometric series

**Quantum algebra:**
- U$_q(sl_2)$, U$_q(\mathfrak{sl}_2)$, $\mathcal{O}_q$, $\mathcal{O}$

**Leonard pairs**
Comment on linear algebra

Let $V$ denote a vector space over $F$ with finite positive dimension.

Assume $A \in \text{End}(V)$ is diagonalizable.

Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of $A$. So

$V = \bigoplus_{i=0}^d V_i \quad \text{(dir sum)}$

For $0 \leq i \leq d$ let $\lambda_i$ denote the eigenvalue of $A$ for $V_i$.

For $0 \leq i \leq d$ define $E_i \in \text{End}(V)$ such that

\[
(E_i - I) V_i = 0
\]

\[
E_0 V_i = 0 \quad \forall j \neq i \quad (0 \leq i \leq d)
\]

So $E_i$ is the projection onto $V_i$. 
Observe

\[ E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq d) \]

\[ I = \sum_{i=0}^{d} E_i \]

\[ A = \sum_{i=0}^{d} \theta_i E_i \]

\[ \begin{align*}
E_i &= \prod_{0 \leq k \leq i} \frac{A - \theta_k I}{\theta_i - \theta_k} & (0 \leq i \leq d) \\
V_i &= E_i V & (0 \leq i \leq d)
\end{align*} \]

Let \( M \) denote the subalgebra of \( \text{End}(U) \) generated by \( A \), then each of

\[ \{ A \} \{ i \geq 0 \} \quad \{ E_i \} \{ i \geq 0 \} \]

is a basis for the \( K \)-vector space \( M \).

Moreover

\[ \prod_{i=0}^{d} (A - \theta_i I) = 0. \]

We call \( \{ E_i \} \{ i \geq 0 \} \) the primitive idempotents of \( A \).