Continue to discuss the split decomp of a T0 system

\( F = \text{any field} \)

\( V = \text{vector space}/IF \text{ with finite pos dim} \)

Fix T0 system in \( V \):

\[ F = (A; \{ E_{i}^{d}, A_{i}^{*}; \{ E_{i}^{*}; i = 0 \} ) \]

Recall the split decomp \( E_{i}^{d}, A_{i}^{*}; \{ E_{i}^{*}; i = 0 \} \):

\[ U_{i} = (E_{i}^{d} + \cdots + E_{i}^{*}) \cap (E_{i}^{d} + \cdots + E_{i}^{*}) \quad o \leq i \leq d \]

Recall 

\[ \rho_{i} = \dim U_{i} \]

Next goal: show \( \rho_{i+1} = \rho_{i} \quad 1 \leq i \leq d/2 \)

DEF. 7.4 Set

\[ R = A - \sum_{h=0}^{d} \theta_{h} F_{h} \]

\[ L = A^{*} - \sum_{h=0}^{d} \theta_{h}^{*} F_{h} \]
LEM 7.5. \( F_n \) 0 is defined as follows:

\[ R = A - \sigma \cdot I, \quad L = A^* - \sigma^* I \]

Proof: Recall \( F_n \) is projection onto \( U_n \) for \( \sigma \neq 0 \).

COR 7.6. \( F_n \) 0 is defined as:

\[ R U_i \leq U_{ir} \]
\[ L U_i \leq U_{ir} \]

Proof: By L7.5 and M67 (ii) \( \square \)
LEMMA 7.7 \[ Fa \quad 0 \leq i \leq j \leq d \]

Let \( f: \mathbb{R}^j \rightarrow \mathbb{R}^i \) be a linear map.

\[ \begin{align*}
  &u_i \rightarrow u_j \\
  &v \rightarrow R^{j-i}v
\end{align*} \]

is injective if \( i+j \leq d \), a bijection if \( i+j = d \),
and surjective if \( i+j > d \).

The map

\[ \begin{align*}
  &u_j \rightarrow u_i \\
  &v \rightarrow L^{j-i}v
\end{align*} \]

is an injection if \( i+j \geq d \), a bijection if \( i+j = d \),
and surjective if \( i+j < d \).

\[ \text{[Caution: above maps are inverses, even if } i+j = d \text{] } \]

PF. Consider \( R^i \).

Case \( i+j \leq d \):
Given $v \in U_i$ such that $R^{\alpha - i} v = 0$ show $v = 0$.

\[ 0 = R^{\alpha - i} v \]

\[ = (A - \omega_{\alpha - i} I) \cdots (A - \omega_{\alpha - i} I) (A - \omega_{\alpha - i} I) v \]

So

\[ v \in E_1 v + E_{\omega_1} v + \cdots + E_{\omega_{\alpha - i}} v \]

\[ \subseteq E_0 v + \cdots + E_{\omega_{\alpha - i}} v \]

Also

\[ v \in U_i \]

\[ \subseteq U_0 + \cdots + U_i \]

\[ = E_0^x v + \cdots + E_i^x v \]

So

\[ v \in (E_0^x v + \cdots + E_i^x v) \cap (E_0 v + \cdots + E_{\omega_{\alpha - i}} v) \]

\[ = 0 \quad \text{(by LLL applied to $\omega$)} \]

Case 1: $\alpha = \omega_i$. $U_i$, $U_j$ have same dim so abone

\[ 1 \geq b \cdot j \]
Case $i+1 \geq d$:

Given $w \in U_i$ find $v \in U_i$ s.t. \( R^{d-i} v = w \)

Consider map

\[
U_{d-2} \rightarrow U_2 \\
\hspace{1cm} u \mapsto R^{d-2} u
\]

This is a bijection.

So \( \exists u \in U_{d-2} \) s.t. \( R^{d-2} u = w \)

Define \( v = R^{d-2-i} u \)

Then \( v \in U_i \) and \( R^{d-i} v = w \).

The proof is similar. \( \Box \)
LEM 78 \[ \pi c \leq \pi e \leq 1/2 \]

Proof: The map

\[ U_{\pi e} \to U_{\pi c} \]

\[ v \to RV \]

is arising by LTT so

\[ \dim U_{\pi c} \leq \dim U_{\pi e} \]

\[ \pi c \leq \pi e \]

\[ \square \]
Next goal: The brahmeton diagram

**Notation**

Given a decom $\{V_i, \beta \} \subseteq V$,

Represent it by a dotted line segment

\[ V_0 \quad V_1 \quad V_2 \quad \ldots \quad \cdots \quad V_{d-1} \quad V_d \]

Given two decomps of $V$:

$\{V_i, \beta \} \subseteq V$

\[ W_0 \quad W_1 \quad W_2 \quad \ldots \quad \cdots \quad W_{d-1} \quad W_d \]

means

\[ \sum_{n=0}^{d} V_n = \sum_{n=0}^{d} W_n \quad \text{for} \quad 0 \leq \varepsilon \leq d \]
Recall that $\mathbf{E}$-split decomposition satisfies

$$U_0 + U_1 + \cdots + U_i = E_0^k V + \cdots + E_i^k V$$

$$U_i + U_{i+1} + \cdots + U_d = E_i V + \cdots + E_d V$$

Corresponding diagram 15:

Apply this to $\mathbf{E}^\Psi$ to get:

Other relations of $\mathbf{E}$ give similar diagrams.
Notation

Let \( \{e_i\}_{i=0}^d \) denote a sequence of positive integers whose sum is \( \dim V \).

A flag \( mV \) of shape \( \{e_i\}_{i=0}^d \) is a nested sequence of subspaces

\[ V_0 \leq V_1 \leq \ldots \leq V_d \]

such that

\[ \dim V_i = e_0 + e_1 + \ldots + e_i \quad \text{for } 0 \leq i \leq d. \]

So \( V_d = V \).

Examples: Let \( \{W_i\}_{i=0}^d \) denote a decomposition of \( V \)

define \( e_i = \dim W_i \) for \( 0 \leq i \leq d \).

Define

\[ V_i = W_0 + \ldots + W_i \quad \text{for } 0 \leq i \leq d. \]

Then \( \{e_i\}_{i=0}^d \) is a flag \( mV \) of shape \( \{e_i\}_{i=0}^d \).

Given two flags \( mV \), denote

\[ \mathcal{E} \{e_i\}_{i=0}^d, \quad \mathcal{E} \{e_i\}_{i=0}^d \]

call these opposite whenever \( \mathcal{E} \) decomposes \( \{e_i\}_{i=0}^d \).

\( \mathcal{E} \) is.
\[ V_i = W_0 + \ldots + W_i \quad (0 \leq i \leq d) \]
\[ V_i^\prime = W_i + \ldots + W_d \]

In this case,

\[ W_i = V_i \wedge V_{i+1} \quad (0 \leq i \leq d) \]

\[ V_i \wedge V_{i+1} = 0 \quad \text{if } i < d \quad (0 \leq i \leq d) \]

So the decompo \( \{ W_i \}_{i=0}^d \) is determined by

the part of opp flag: "associated decompo"

**Def 79.** For our IP system \( E \) we now define 4 flags in \( V \), denoted \([0], [0], [0^*], [0^*] \)

Each flag has shape \( \{ p_i \}_{i=0}^d \)

<table>
<thead>
<tr>
<th>Flag</th>
<th>1st component</th>
</tr>
</thead>
<tbody>
<tr>
<td>[0]</td>
<td>( E_0 V + \ldots + E_i V )</td>
</tr>
<tr>
<td>[0]</td>
<td>( E_0 V + \ldots + E_0 i V )</td>
</tr>
<tr>
<td>[0^*]</td>
<td>( E_0^* V + \ldots + E_i^* V )</td>
</tr>
<tr>
<td>[0^*]</td>
<td>( E_0^* V + \ldots + E_0^* i V )</td>
</tr>
</tbody>
</table>
LEM 80  The four flaps in Oct 79
are mutually opposite.

pf  Show \([a^*], [0]\) are opp.

Consider \(E\)-split decom \(\{u_i\}_{i=0}^d 1V\)

\[ \text{Favored} \]
\[ \text{in component of } [a^*] = E_0^*V + \cdots + E_k^*V \]
\[ = u_0 + t\alpha \]
\[ \text{in component of } [0] = E_0V + \cdots + E_0V \]
\[ = u_d + \cdots + u_d \]

Rest of pf is same.

Notation  Given ordered pair of dist
flaps in Oct 79
\([a], [\beta]\)

Let \([a, \beta]\) denote the associated decom \(1V\).
We have

<table>
<thead>
<tr>
<th>$[0; 0]$</th>
<th>$E_iV$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[0^<em>, 0^</em>]$</td>
<td>$E_i^*V$</td>
</tr>
<tr>
<td>$[0^*, 0]$</td>
<td>$(E_0^*V + \cdots + E_i^*V) \cap (E_iV + \cdots + E_0V)$</td>
</tr>
<tr>
<td>$[0^<em>, 0^</em>]$</td>
<td>$(E_0^*V + \cdots + E_i^*V) \cap (E_iV + \cdots + E_0V)$</td>
</tr>
<tr>
<td>$[0^<em>, 0^</em>]$</td>
<td>$(E_0^*V + \cdots + E_{i-1}^*V) \cap (E_iV + \cdots + E_0V)$</td>
</tr>
<tr>
<td>$[0^<em>, 0^</em>]$</td>
<td>$(E_0^*V + \cdots + E_{i-1}^*V) \cap (E_iV + \cdots + E_0V)$</td>
</tr>
</tbody>
</table>

We now describe the actions of $A, A^*$ on the above decomp.
Thm 31. Let \( \{ w_i \} \) denote any one of the above 6 decomps of \( V \).

Then for \( 0 \leq i \leq 6 \), the action of \( A, A^* \) on \( W_i \) is:

<table>
<thead>
<tr>
<th>( A ) action</th>
<th>( A^* ) action</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0, 0]) ( (A - \Theta; I) w_i = 0 ) ( A^* w_i \leq w_i + r w_i + w_{in} )</td>
<td></td>
</tr>
<tr>
<td>([0^<em>, 0]) ( A W_i \leq w_i + r w_i + w_{in} ) ( (A^</em> - \Theta; I) w_i = 0 )</td>
<td></td>
</tr>
<tr>
<td>([0^<em>, 0^</em>]) ( (A - \Theta; I) w_i \leq w_{in} ) ( (A^* - \Theta; I) W_i \leq w_i )</td>
<td></td>
</tr>
<tr>
<td>([0^<em>, 0]) ( (A - \Theta; I) w_i \leq w_{in} ) ( (A^</em> - \Theta; I) W_i \leq w_{in} )</td>
<td></td>
</tr>
<tr>
<td>([0^<em>, 0]) ( (A - \Theta; I) w_i \leq w_{in} ) ( (A^</em> - \Theta; I) W_i \leq w_{in} )</td>
<td></td>
</tr>
<tr>
<td>([0^<em>, 0]) ( (A - \Theta; I) w_i \leq w_{in} ) ( (A^</em> - \Theta; I) W_i \leq w_{in} )</td>
<td></td>
</tr>
</tbody>
</table>

Proof: Rows \([0, 0]\) and \([0^*, 0^*]\) are from left TD system.

Row \([0^*, 0]\) is from \( \Theta \rightarrow (ii) \).

To get remaining rows, apply \( \Theta \) to relations of \( \Theta \).