Lecture 35  Wed Dec 4  12/4/13

Recall $U_q = U_q(sl_2)$

$U_q$-module $V = V_{a,1}$

dual space $V^*$ is $U_q$-$module$

LEM 43 Let $\{V_i\}_{i=0}^d$ denote a decomp $f^a V$ or $V^*$

Then $\{V_i\}_{i=0}^d$ is equal to $[y]$ iff both

(i) $n_x$ is raising $f \{V_i\}_{i=0}^d$

(ii) $n_z$ is lowering $f \{V_i\}_{i=0}^d$

pf Assume (i), (ii) then

$n_z V_0 = 0$ \hspace{1cm} $n_x V_i \leq V_{i+1}$ \hspace{1cm} $0 \leq i \leq d$

and

$n_x V_d = 0$ \hspace{1cm} $n_z V_i \leq V_{i-1}$ \hspace{1cm} $1 \leq i \leq d$

Now $\{V_i\}_{i=0}^d$ is $[y]$ by L42 (i), (ii), (iv).

Converse is similar $\square$
LEM 44 \( \text{Let } \{V_i\}_{i=0}^d \text{ denote a decom of } V \text{ or } V^* \text{. Then } \{V_i\}_{i=0}^d \text{ is equal to } [y] \text{ iff all } \)

(i) \( x \text{ is quasi-lowering on } \{V_i\}_{i=0}^d \)

(ii) \( \gamma \text{ is diagonal on } \)

(iii) \( \varepsilon \text{ is quasi-rising on } \)

\[ pf \begin{array}{c}
\Rightarrow \\
\text{by } \ref{lem:9} \text{ or } \ref{lem:40} \\
\Leftrightarrow \text{ use } \ref{lem:42} (i), (iii)
\end{array} \]

Subspace \( V_0 \) is inv under \( x, y \)

\( n_2 \) is sc mult \( y (1-x) y \)

\( V_0 \) is inv under \( n_2 \)

\( n_2 \) is nilp and \( \dim V_0 = 1 \) so

\( n_2 V_0 = 0 \)

\( \sim n_2 V_d = 0 \)

\( F_0, 0 \leq i \leq d-1, \)

\( \sim V_i \leq V_i + V_{i+1}, \quad y V_i \leq V_i, \quad y V_i \leq V_i \)

\( \text{so } y V_i \leq V_i + V_{i+1} \)

\( n_2 = \text{ sc mult } y (1-x) \)

\( n_2 V_i \leq V_i + V_{i+1} \)
\(\forall x \quad n(x \in \mathbb{N}) \quad \sum_{i=1}^{d} v_i = 1 \quad 0 \leq g \leq d \)

So

\(n_X v_i \leq v_{im} \quad 0 \leq i \leq d\)

Now by L92 (i), (ii)

\(\{ v_i \}_{i=0}^d \in [G] \).

Q
Notation

A flag on $V$ is a sequence of subspaces $\{U_i\}_{i=0}^d$ of $V$ s.t.
$U_i \subseteq U_{i+1}$ for $i \leq d$ and $\dim U_i = i$.

So $U_d = V$.

Given decomposition $\{V_i\}_{i=0}^d$ of $V$, define

$U_i = V_0 + \cdots + V_i$ for $i \leq d$.

Then $\{U_i\}_{i=0}^d$ is flag on $V$, said to be induced by $\{V_i\}_{i=0}^d$.

Given two flags $\{U_i\}_{i=0}^d$ and $\{U'_i\}_{i=0}^d$ on $V$.

Call them opposite whenever

$U_i \cap U'_j = 0$ if $i \neq j$ (or $i = d$).

The above flags are opposite iff $\exists$ decomposition $\{V_i\}_{i=0}^d$ of $V$

that induces $\{U_i\}_{i=0}^d$ and whose inverse $\{V_i\}_{i=0}^d$ induces $\{U'_i\}_{i=0}^d$.

In this case

$V_i = U_i \cap U'_i$ for $0 \leq i \leq d$. 

LEM 45. For $0 \leq i \leq d+1$,

\[ n^i_x V \text{ is the sum of components } z_{i+1},...,d \text{ of } \text{decomp } [y] \]

and the sum of components $0,1,...,d-i$ of $\text{decomp } [z]$.

\[ (+CP) \]

pf. Let $\{y_j\}_{j=0}^d$ denote $\text{decomp } [y]$.

\[ V = \sum_{j=0}^d y_j \quad ds \]

By LEM 38,

\[ n_x^i V_j = \begin{cases} 
V_j & 0 \leq j \leq d \quad V_{d+1} = 0 
\end{cases} \]

So,

\[ n_x^i V = V_0 + V_1 + \cdots + V_d \]

Last assertion sim.
Cor 46 \[ \bigwedge^{i} V \text{ has dim } d-i \]

for \( 0 \leq i \leq d \)

\[ (+ CP) \]

pf by L45

Cor 47 the sequence

\[ \{ \bigwedge^{d-i} V \}_{i=0}^{d} \]

is a flag on \( V \)

\[ (+ CP) \]

pf by Cor 46 + const.
LEM 48. In each row of the table below, we give a decomposition of $V$ and the induced flag on $V$.

<table>
<thead>
<tr>
<th>Decomposition of $V$</th>
<th>Induced Flag on $V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[x]$</td>
<td>${ n^{d-i}V }_{i=0}^d$</td>
</tr>
<tr>
<td>$[x]^\mathbb{N}$</td>
<td>${ n^{d-i}V }_{i=0}^d$</td>
</tr>
</tbody>
</table>

$(+ CP)$
LEM 49  \textbf{the flags}

\[
\{ n_{x-d-1}^d V \}_i \subseteq \{ n_{y-d} V \}_i \subseteq \{ n_{z}^d V \}_i \subseteq \{ n_{x-d-1}^d V \}_i
\]

are mutually opposite.

\textbf{pf.}  By L48 and of opposite flags.

\begin{center}
\begin{tabular}{ccc}
\textbf{Decomp of } V & \textbf{ith comp} \\
$[x]$ & $n_{y-d} V \cap n_{z}^d V$ \\
$[x]^{i \forall}$ & $n_{y}^d V \cap n_{x-d-1}^d V$
\end{tabular}
\end{center}

\textbf{pf.}  Use L45
Given decompostion of \( V \): \( \mathfrak{F}_{V_i} \mathfrak{F} \)

Represented by line segment

\[ V_0 \quad V_1 \quad V_2 \quad \cdots \quad V_{a+b} \]

Given 2nd decompostion of \( V \): \( \mathfrak{F}_{V_i'} \mathfrak{F} \)

\[ V_0 \quad V_1 \quad V_2 \quad \cdots \quad V_{a+b} \]

Means

\[ V_0 + V_i = V_0' + V_i' \]

\[ \forall i \in \mathbb{N} \]

\[ \mathfrak{F} \text{ is induced by } \mathfrak{F}_{V_i'} \mathfrak{F} \]

\[ = \cdots \cdots \quad \mathfrak{F}_{V_i'} \mathfrak{F} \]
The diagram

\[ \begin{array}{c}
\text{The diagram} \\
\times \\
\rightarrow \rightarrow \\
\end{array} \]

means: displayed decom is eigenspace decom of \( x \),
with eigenvalues

\[ \begin{array}{c}
\text{get } \lambda_1, \lambda_2, \ldots, \lambda_d \\
\end{array} \]

The chy-module \( V \):
the Aug $\bigoplus_{n_2} \bigoplus_{u_1} \bigoplus_{\cdots}$
Action of $\pi_2$ on $V_1$
the action of $y$ on $V$:
LEM 51 \quad F_{n} \quad 0 \leq i \leq d + 1.

\text{n}_{x}^{i}V \quad \text{is the unique} \quad (d-i+1)-\text{dim} \quad \text{subspace of} \quad V

\text{that is} \quad n_{x} \quad \text{under} \quad y_{i+2}.

pf \quad \text{n}_{x}^{i}V \quad \text{has desired features by above disc.}

Now let \( W = (d-i+1)-\text{dim} \quad \text{subspace of} \quad V \) \quad \text{that is} \quad n_{x} \quad \text{under} \quad y_{i+2}.

Show \quad W = n_{x}^{i}V

Case \quad i = d + 1:

\[ W = 0 = n_{x}^{i-1}V \]

Case \quad i = i - 1:

\[ W + o \]

Let \( \{ v_{i} \}_{i=0}^{d} \) = \text{decomp} \quad [y_{i}] \quad \text{of} \quad V.

\( y \) is diagonalizable \quad mV, \quad \text{W is} \quad y_{-}mV.

\( y \) is diagonalizable \quad mV

\[ W = \sum_{i} v_{i} \quad S = \{ 0 / o \text{stabilized} \quad v_{i} \in W \} \]

\( s + o \) since \( W + o \)

\[ nx = sc \quad \text{mult} \quad 1 - y_{2} \]

\( W \) is \quad n_{x} \quad \text{under} \quad y_{i+2} \)

\[ nx \quad v_{i} = v_{y_{2}} \quad 0 \leq i \leq d + 1 \]

\( \geq S \rightarrow y_{2} \in S \quad 0 \leq i \leq d + 1 \)
\[ \exists t \ (0 \leq t < d) \ \text{s.t.} \]

\[ S = \{ t, t+1, \ldots, d \} \]

\[ W = \sum_{g=t}^{d} V_g \]

\[ \dim W = d - t \]

\[ d - \text{in} \]

\[ t = i \]

\[ W = \sum_{g=i}^{d} V_g \]

\[ = n x V \]

\[ \square \]