We continue to study the graph $\Gamma = P_D$, a path of length $2D$. $x$ is an end-vertex of $\Gamma$.

Recall $\eta \in \mathbb{C}$ is a $(20+4)$-th root of unity.

$A^* = \text{diag}\left(\theta_0^*, \ldots, \theta_D^*\right)$

$\theta_i^* = \eta^{i+1} + \eta^{-i+1}$ \hspace{1cm} $0 \leq i \leq 0$

Theorem: The eigenvalues of $\Gamma = P_D$ are

$\theta_i = \eta^{i+1} + \eta^{-i+1}$ \hspace{1cm} $(0 \leq i \leq 0)$

For this ordering $A^*$ is a dual adjacency matrix with respect to $x$.

Moreover

$E_i A^* E_i = 0$ \hspace{1cm} $(0 \leq i \leq 0)$

"dual bipartite"
Proof: For the time being let
\[ \Theta \] denote any ordering of the eigenvalues of \( \Gamma \). We saw earlier \( \Theta \geq 0 \).

But \( \Theta + 1 \leq |x| = 0 + 1 \)

So \( \Theta = 0 \)

and \( \dim E_i \nu = 1 \) for \( 0 \leq i \leq D \).

Draw a diagram on the nodes \( 0, 1, \ldots, D \).

For \( 0 \leq i \leq D \) node \( i \) represents \( \Theta_i \) in \( E_i \).

For \( 0 \leq i, j \leq D \) attach nodes \( i, j \) by an arc \( i \rightarrow j \) whenever \( E_i A^* E_j \neq 0 \)

(\( i \rightarrow j \) gets a loop \( i \) whenever \( E_i A^* E_i \neq 0 \))

Note that \( E_i A^* E_j = 0 \) iff \( E_j A^* E_i = 0 \)

Since \( E_i A^* E_j = E_j A^* E_i \)

Therefore the diagram is undirected.
Claim 1. For \(0 \leq t \leq 1\) assume

nodes \(x, y\) are connected by an arc. Then

\[
\theta_i^2 - \beta \theta_i \theta_j + \theta_j^2 = -(9 - q^2)^2
\]

Proof. Consider eq. (i) in Prop 9

\[
o = E_i \left( \text{LHS} - \text{RHS} \right) E_j
\]

\[
= E_i A^* E_j \left( \theta_i^2 - \beta \theta_i \theta_j + \theta_j^2 + (9 - q^2)^2 \right)
\]

\[= 0 \quad \text{must be 0}\]

Claim 2. Each node \(x\) in diagram is connected by an arc to at most 2 nodes in the diagram.

Proof. For each node \(y\) that is connected to node \(x\) by an arc, \(\theta_j\) is a root of the quadratic polynomial

\[
\lambda^2 - \beta \theta_i \lambda + \theta_i^2 + (9 - q^2)^2 = 0
\]
Claim 3: In the diagram, assume node \( i \) is adjacent node \( r, a \) (\( r \neq a \)).

Then

\[ \theta_r - \beta \theta_i + \theta_a = 0 \]

Proof of Claim:

Both

\[ \theta_i^2 - \beta \theta_i \theta_r + \theta_r^2 = - (r - q^{-1})^2, \]

\[ \theta_i^2 - \beta \theta_i \theta_a + \theta_a^2 = - (r - q^{-1})^2. \]

Take the difference:

\[ \theta_r^2 - \theta_a^2 = \beta \theta_i (\theta_r - \theta_a) \]

Hence

\[ (\theta_r - \theta_a)(\theta_r + \theta_a) \]

\[ \theta_r + \theta_a = \beta \theta_i \]

\[ \theta_r + \theta_a = \beta \theta_i \]
So far, the possible diagrams are

\[ \cdots \cdots \cdots \]

In my case, without our ordering \( \theta_i \) satisfies \( \theta_{\alpha} \geq \theta_i \) for all \( i \in D \).

By claim 1

\[ \theta_i^2 - \beta \theta_i \theta_n + \theta_n^2 = -(q-q^2)^2 (1 \leq i \leq 0) \]  

By claim 3

\[ \theta_i - \beta \theta_i + \theta_n = 0 \]  

By (4) and since \( \beta = q + q^{-1} \), there exist \( a, b \in \mathbb{C} \) for any

\[ \theta_i = a q^i + b q^{-i} \]  

(0 ≤ i ≤ 0)
Evaluate (4) using Hess to get

\[ ab = 1 \]

So

\[ a_i = a_0 q^i + a^{-1} q^{-i} \quad (i \leq i \leq 0) \]

All diagonal entries of \( A \) are 0. Therefore

\[ o = \text{trace} (A) \]

\[ = \sum_{i=0}^{0} a_i \]

\[ = a \left( 1 + q + q^2 + \cdots + q^0 \right) + a^{-1} \left( 1 + q^{-1} + q^{-2} + \cdots + q^{-0} \right) \]

\[ = a^{-q^{-0}} \left( 1 + q + q^2 + \cdots + q^0 \right) \]

\[ = \left( a + a^{-q^{-0}} \right) \frac{q^{0+1} - 1}{q - 1} \]

But

\[ q^{0+1} - 1 = -q^1 - 1 \neq 0 \]

So

\[ a + a^{-q^{-0}} = 0 \]

Therefore

\[ a^2 = -q^{-0} = q^2 \]
So \( a = \pm 2 \)

For \( \alpha = 2 \) get same list in reverse order

So wlog

\[ \theta_i = \theta_i^{\text{op}} + \theta_i^{\text{op}} \quad (0 \leq i \leq 0) \]

**Claim 4**  
In the diagram, no loop at \( \theta_0 \) or \( \theta_0 \). Also \( \theta_0, \theta_0 \) are not connected by an arc, provided that \( \alpha \geq 2 \).

**pf cl**  
Use claim 3. One checks

\[ \theta_0 - \beta \theta_0 + \theta_1 \neq 0 \]
\[ \theta_0 - \beta \theta_0 + \theta_0 \neq 0 \]
\[ \theta_1 - \beta \theta_0 + \theta_0 \neq 0 \]

By the above claim the diagram is

So \( E_i A X E_j = 0 \) if \( |i-j| = 1 \) \( (0 \leq i, j \leq 0) \)
Until further notice \( \Gamma = (X, \Omega) \) denotes any connected graph.

To avoid trivialities assume \( |X| \geq 2 \).

Write 
\[
\mathbf{1} = \sum_{x \in X} \mathbf{1}^x \quad \text{"all 1's vector"}
\]

Fix \( x \in X \) and write \( \mathbf{M}^x = \mathbf{M}^x(x) \), \( \mathbf{T} = \mathbf{T}(x) \), \( d = D(x) \).

For \( i \in Z \) define
\[
\Gamma_i^x(x) = \{ y \in X \mid 2(x,y) = i \} \]

So
\[
\Gamma_i^x(x) = \emptyset \text{ if } i < 0 \text{ or } i > d
\]

Also
\[
\mathbf{E} = \mathbf{E}(x) \quad \Gamma_i^x(x) = \Gamma_i(x)
\]

For \( 0 \leq i \leq d \)
\[
\{ \mathbf{y} \in \Gamma_i^x(x) \} \text{ is a basis for } \mathbf{E}^x \mathbf{V}
\]

Define 
\[
\mathbf{A}_i = \mathbf{A}_i^x(x) = |\Gamma_i^x(x)|
\]
So \[ k_i = \dim E_i^* V \]

\[
\begin{align*}
k_0 &= 1 \\
k_i &= k_i(x) = \text{valency of } x \\
|\chi| &= \sum_{i=0}^{d} k_i
\end{align*}
\]

**DEF 12** \( \Gamma \) is said to be **distance-regular** with respect to \( x \) whenever for \( 0 \leq i \leq d \) and \( y \in \Gamma_i(x) \)

\[
\begin{align*}
c_i &= \left| \Gamma(y) \cap \Gamma_{i+1}(x) \right| \\
a_i &= \left| \Gamma(y) \cap \Gamma_i(x) \right| \\
b_i &= \left| \Gamma(y) \cap \Gamma_{i-1}(x) \right|
\end{align*}
\]

The intersection numbers \( \Gamma \) wrt \( x \)

Note: \( a_i, b_i, c_i \) are equal