Recall our connected graph $\Gamma = (X, \mathcal{R})$, $|X| = 2$

Fix $x \in X$, write $M^x = M(x)$, $T = T(x)$, $d = d_x$

Until further notice assume that $\Gamma$ is distance-regular with respect to $x$.

Observe

- $c_i \neq 0$, $1 \leq i \leq d$, $c_0 = 0$
- $b_i \neq 0$, $0 \leq i \leq d + 1$, $b_d = 0$
- $a_0 = 0$, $q = 1$

**LEM 13**

(i) $k_i c_i = k_i b_i$, $1 \leq i \leq d$

(ii) $k_i = \frac{b_0 b_1 \ldots b_{i-1}}{c_1 c_2 \ldots c_{i-1}}$, $0 \leq i \leq d$

**pf** (i) Count in two ways the edges between $\Gamma_i (x)$ and $\Gamma_{i-1} (x)$.

(ii) By (i) and induction on $i$.
For $0 \leq i \leq d$ define

\[ \Pi_i = E_i^* \Pi \]

\[ = \sum_{\gamma \in \Gamma_i(\infty)} \gamma \]

So

\[ \Pi_0 = x, \]

\[ \langle \Pi_i, \Pi_j \rangle = \delta_{ij} \Pi_i \]

\[ 0 \leq i, j \leq d \]

\[ E_i^* \Pi_j = \delta_{ij} \Pi_i \]

\[ 0 \leq i, j \leq d \]

We now consider the action of $A$ on $\{ \Pi_i : i = 0 \}$.
LEM 14

(i) \( A \Pi_0 = \Pi_0 \)

(ii) \( A \Pi_i = b_{in} \Pi_{in} + a_i \Pi_i + c_{in} \Pi_{in} \quad 1 \leq i \leq d^+ \)

(iii) \( A \Pi_d = b_{dr} \Pi_{dr} + a_d \Pi_d \)

\( \rho^f \) (iii) For \( 1 \leq i \leq d^+ \),

\[ A \Pi_i = A \sum_{\gamma \in \Gamma_i(x)} \frac{\hat{Z}}{\gamma} \]

\[ = \sum_{\gamma \in \Gamma_i(x)} \sum_{\tilde{z} \in \Gamma(\gamma)} \frac{\hat{Z}}{\tilde{z}} \]

\[ = \sum_{\tilde{z} \in \chi} \frac{\hat{Z}}{|\Gamma(\tilde{z}) \cap \Gamma_i(x)|} \]

\[ = b_{in} \sum_{\tilde{z} \in \Gamma_{in}(x)} \frac{\hat{Z}}{\tilde{z}} + a_i \sum_{\tilde{z} \in \Gamma_i(x)} \frac{\hat{Z}}{\tilde{z}} + c_{in} \sum_{\tilde{z} \in \Gamma_{in}(x)} \frac{\hat{Z}}{\tilde{z}} \]

\[ = b_{in} \Pi_{in} + a_i \Pi_i + c_{in} \Pi_{in} \]

(i), (iii) Simlar.

\( \square \)
LEM 15. The vectors \( \{ \tilde{e}_i \}_{i=0}^d \) form a basis for the primary \( T \)-module. Relative this basis,

\[
A = \begin{pmatrix}
  a_0 & b_0 \\
  c_1 & a_1 & b_1 \\
  & \ddots & \ddots & \ddots \\
  & & & c_d & a_d \\
  & & & & 0
\end{pmatrix}
\]

\( E_i : \text{diag}(0, \ldots, 0, i, 0, \ldots, 0) \quad (0 \leq i \leq d) \)

pf. Let \( W \) denote the subspace of \( V \) spanned by \( \{ \tilde{e}_i \}_{i=0}^d \). By LEM 14, \( AW = W \).

We saw \( E_i^* P_i = \delta_{ij} P_j \quad (0 \leq i, j \leq d) \).

\( E_i^* W = W \) for \( 0 \leq i \leq d \). So \( W \) is a \( T \)-module.

Let \( \tilde{W} \) denote the primary \( T \)-module. Show \( W = \tilde{W} \).

By construction \( x^* = 1 \circ e \in W \). Also \( x^* \in \tilde{W} \).

So \( W \cap \tilde{W} = 0 \).
$W/\sim$ is a non-$T$-module contained in $\tilde{W}$. $T$-module $\tilde{W}$ is irreducible, so

$W/\sim = \tilde{W}$, i.e. $\tilde{W} \leq W$

By construction

$\dim W = d + t$

We saw earlier

$\dim \tilde{W} \geq d + t$

So $W = \tilde{W}$. $\square$
We now bring in some polynomials in one variable.

Let $\lambda$ be indeterminate.

Let $C[\lambda] = C$-algebra of polynomials in $\lambda$ that have all coeffs in $C$.

For $0 \leq i \leq m$, define $f_i \in C[\lambda]$ by

\[ f_0 = 1, \quad f_1 = \lambda \]

\[ \lambda f_i = b_{i-1} f_{i-1} + a_i f_i + c_i f_{i+1} \quad \text{for} \quad 1 \leq i \leq n - 1 \]

\[ \lambda f_d = b_{d-1} f_{d-1} + a_d f_d + \frac{f_{d+1}}{\xi_{c_2} \cdots c_{d}} \]

Observe that $f_d$ for $0 \leq i \leq d$

$f_i$ has degree $i$, and coef $f_1$ is $\frac{1}{\xi_{c_2} \cdots c_{d}}$

Also

$f_{d+1}$ is monoic with degree $d+1$. 

LEM 16

(i) \( f_i(A) x^i = \Pi_i \) \( \text{osized} \)

(ii) \( \text{fddn}(A) x = 0 \)

pf Compare the def of \( f_i \) with LEM 14.

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Recall that \( M \) is the subalgebra of \( \text{Mat}_N(C) \) generated by \( A \).

LEM 17

(i) the primary \( T \)-module is \( M x^i \)

(ii) For the action of \( A \) on \( M x^i \), \( \text{fddn} \) is both the min poly. and char poly.

pf (i) By LEM 16 \( M x^i \) has basis \( \{ x_i, x_i^2, \ldots \} \).

By LEM 15.

(iii) \( \text{fddn}(A) M x^i = M \text{fddn}(A) x^i = 0 \).

So min poly of \( A \) in \( M x^i \) divides \( \text{fddn} \).

\( A \) is diagonalizable on \( M x^i \) so on \( M x^j \).

min poly of \( A \) = char poly of \( A \).
chance that $A$ in $M_n$ has degree $d_1$.

Result follows.

DEF 18. Let $\{\theta_i\}_{i=0}^d$ denote the roots of \( f \).

[These roots are among the eigenvalues of $$.

Call $\{\theta_i\}_{i=0}^d$ the primary eigenvalues of $f$.

with respect to $x$.}
LEM 19 For any eigenvalue \( \theta_i \) of \( \Gamma \),

(i) Assume \( \theta_i \) is primary, then \( E_i \mathbf{x}^\theta \) is a basis for \( E_i \mathbf{M}^\theta \).

(ii) Assume \( \theta_i \) is not primary, then \( E_i \mathbf{x}^\theta = 0 \).

pf (i) By def \( E_i \mathbf{M}^\theta \neq 0 \), \( E_i \) is a prim independent.

\[
E_i \mathbf{M}^\theta = CE_i.
\]

So \( E_i \mathbf{M}^\theta = CE_i \mathbf{x}^\theta \neq 0 \).

(ii) By def of primary, \( E_i \mathbf{M}^\theta = 0 \). 

\[ \square \]

LEM 20 \( \{ E_i \mathbf{x}^\theta \}_{i=0}^d \) is an or-meg basis for \( \mathbf{M}^\theta \).

pf By L19 and since

\[
\mathbf{M}^\theta = \sum_{i=0}^d E_i \mathbf{M}^\theta \quad \text{(ods)}
\]

\[ \square \]
DEF 21 For $0 \leq i \leq d$ define

$$A_i = E_i \cdot k$$

$$m_i = \| A_i \|^2$$

Note

$$\hat{x} = \sum_{i=0}^{d} A_i^*$$

$$m_i \neq 0 \quad 0 \leq i \leq d$$

$$1 = \sum_{i=0}^{d} m_i$$

We have seen that both

$$\{ A_i \}_{i=0}^{d}, \quad \{ A_i^* \}_{i=0}^{d}$$

are orthogonal bases for $M^\infty$. We now consider how these bases are related.
LEMMA 22: For $0 \leq i \leq d$

$$\langle \Pi^i, \Pi^i_z \rangle = f_i(\theta_1) m_i$$

**Proof**

$$\langle \Pi^i, \Pi^i_z \rangle = \left\langle \begin{array}{c} f_i(A) x_i \\
E_2 \end{array} \right| \begin{array}{c} E_1 \hat{x} \\
E_2 \end{array} \right\rangle$$

$$= \left\langle \underbrace{E_2 \ f_i(A) x_i}_{\Pi} \right| \begin{array}{c} E_2 \hat{x} \\
E_2 \end{array} \right\rangle$$

$$= f_i(\theta_1) \parallel E_2 \hat{x} \parallel$$

$$= f_i(\theta_1) m_i.$$

$\square$
We now give the transition matrices between our bases $\mathcal{M}$.

**LEM 23**

For $0 \leq j \leq d$

\[ I_j = \sum_{i=0}^{d} f_j(\theta_i) I_i^* \]

\[ I_j^* = \sum_{i=0}^{d} \sum_{k=i}^{d} \frac{f_i(\theta_j)}{k^i} I_i^* \]

**pf (i)**

\[ I_j = f_j(A)^x \]

\[ = I f_j(A) I^x \]

\[ = \sum_{i=0}^{d} f_j(A) E_i^x \]

\[ = \sum_{i=0}^{d} f_j(A) E_i I_i^x \]

\[ = \sum_{i=0}^{d} f_j(\theta_i) E_i I_i^x \]
(iii) Use LEM 22. Write

\[ \Pi^k \equiv \sum_{h=0}^{d} \lambda_h \Pi_h \quad \lambda_h \in \mathbb{C} \]

For \(0 \leq i \leq d\)

\[ \langle \Pi_i, \Pi^k \rangle = \langle \Pi_i, \sum_{h=0}^{d} \lambda_h \Pi_h \rangle \]

\[ = \lambda_i k_i \]

By LEM 22

\[ \langle \Pi_i, \Pi^k \rangle = f_i(c_0)/m_i \]

So

\[ \lambda_i = \frac{f_i(c_0)/m_i}{k_i} = \lambda_i \]

\[ \square \]
The polynomials \( \{ f_i \}_{i=0}^d \) are "orthogonal" in the following sense.

**LEM 24** \( \forall r, s, t \in \mathcal{D} \)

(i) \[
\sum_{i=0}^{d} f_r(\theta_i) f_s(\theta_i) m_i = \delta_{rs} kr
\]

(ii) \[
\sum_{i=0}^{d} f_r(\theta_i) f_t(\theta_i) ki_i = \delta_{rt} m_i
\]

**pf (i)**

\[
\delta_{rs} kr = \langle \Pi_r, \Pi_s \rangle
\]

\[
= \langle \sum_{i=0}^{d} f_r(\theta_i) \Pi_i, \sum_{j=0}^{d} f_s(\theta_j) \Pi_j \rangle
\]

\[
= \sum_{i=0}^{d} f_r(\theta_i) f_s(\theta_i) m_i
\]
\[ \delta_{\text{rad } m_{r}} = \left\langle \Pi_{r}^{x}, \Pi_{a}^{x} \right\rangle \]

\[ = \left\langle \sum_{i=0}^{d} \frac{f_{i}(\theta_{r}) m_{r}}{k_{i}} \Pi_{i}, \sum_{g=0}^{d} \frac{f_{g}(\theta_{a}) m_{a}}{k_{g}} \Pi_{g} \right\rangle \]

\[ = \sum_{i=0}^{d} \frac{f_{i}(\theta_{r}) f_{i}(\theta_{a}) m_{r} m_{a}}{k_{i}^{2}} \]

Result follows.

\[ \square \]
DEF 25. \( \Gamma \) is said to be

**distance-regular** whenever \( tx \in X \)

(i) \( \Gamma \) is distance-regular with respect to \( x \);

(ii) The intersection numbers of \( \Gamma \) with respect to \( x \) do not depend on \( x \).