We now consider a family of distance-regular graphs called the hypercubes.

**DEF 26** Fix an integer $D \geq 1$. The graph $Q_D$ has vertex set

$$X = \{ x \mid x \in \{1,-1\}^D \}$$

and edge set

$$E = \{ \{ x, y \} \mid x, y \in X, \ x \neq y \text{ differ in exactly one coordinate} \}$$

Call $Q_D$ the $D$-cube or hypercube. $Q_2$ is also called the Hamming graph $H(2,2)$. 

Basic facts about $Q_0$.

- The graph $Q_0$ is connected. For vertices $x, y$ of $Q_0$ the distance $d(x, y) = \#$ of coordinates at which $x, y$ differ.

Moreover, the diameter of $Q_0$ is 3.

- The graph $Q_0$ is bipartite. The bipartition is $X = X^+ \cup X^-$.

$X^+ = \{ x \in X \mid x \text{ has even weight} \}$

$X^- = \{ x \in X \mid x \text{ has odd weight} \}$

where the weight of $x$ is the number of positive coordinates of $x$.

- The graph $Q_0$ is distance-regular, with intersection numbers $e_0 = 1$, $a_0 = 0$, $b_0 = 3 - i$, $0 \leq i \leq 0$.

The valencies are

$$k_0 = \begin{pmatrix} \rho \\ i \end{pmatrix}, \quad 0 \leq i \leq 0$$
Consider the eigenvalues of $\Phi_1$.

Find pattern

\[
\begin{array}{c|ccc}
\text{equal} & 1 & -1 \\
\text{mult} & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\text{equal} & 2 & 0 & -2 \\
\text{mult} & 1 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\text{equal} & 3 & 1 & -1 & -3 \\
\text{mult} & 1 & 3 & 3 & 1 \\
\end{array}
\]

Not hard to guess

ex. the eigenvalues of $\Phi_2$ are

$\lambda_i = \varnothing - 2\iota$

$\lambda_i \in \mathbb{C}$

with $\lambda_i$ having multiplicity \( (\iota) \)

for $i \in \mathbb{R}$.
Until further notice for $x \in X$ and write $M^x = M^x(x)$, $T = T(x)$, etc.

WLOG $x = \frac{11 \ldots 1}{0}$

**DEF 27** For $Q_0$ define $R, L \in \text{Mat}_X(Q)$ by

$$R = \sum_{i=0}^{d-1} E_i^x A E_i^x$$

"raising matrix"

$$L = \sum_{i=1}^{d} E_i^x A E_i^x$$

"lowering matrix"

We observe

$$\overline{R} = R, \quad \overline{L} = L, \quad R^t = L$$

$$A = R + L$$

$$RE_i^x V \leq E_i^x V \quad 0 \leq i \leq 0$$

$$LE_i^x V \leq E_i^x V \quad 0 \leq i \leq 0$$
DEF 28 For $q_0$ Define

$$A^x = \sum_{i=0}^{0} e_i^x \mathbf{E}_i^x$$

where

$$e_i^x = 0 - 2i \quad \quad 0 \leq i \leq 0.$$ 

---

We will show that $A^x$ is a dual adjacency matrix w.r.t $x$ and $\{e_i^x\}_{i=0}^{0}$.  

LEM 29 \hspace{1cm} \text{For } \varphi_0 \text{ pick any } y_1, z \in X \text{ such that }
\begin{align*}
\varphi(y_1, z) &= 2 \quad \text{and} \quad \varphi(x_1 y_1) = \varphi(x_1 z).
\end{align*}

(i) \hspace{1cm} \exists \text{ unique vertex } u \in X \text{ st. }
\begin{align*}
\varphi(u, y_1) &= 1, \\
\varphi(u, z) &= 1, \\
\varphi(u, x) &= \varphi(x_1 y_1) - 1.
\end{align*}

(ii) \hspace{1cm} \exists \text{ unique vertex } v \in X \text{ st. }
\begin{align*}
\varphi(v, y_1) &= 1, \\
\varphi(v, z) &= 1, \\
\varphi(v, x) &= \varphi(x_1 y_1) + 1.
\end{align*}

pf \hspace{1cm} \text{Routine}

Illustration for } n = 3
LEM 30  
We have

(i) \[ LR - RL = A^* \]
(ii) \[ A^* L - LA^* = 2L \]
(iii) \[ A^* R - RA^* = -2R \]

pf (i)  
For \( y, z \in X \) we compute the \((y, z)\)-entry of each side.

<table>
<thead>
<tr>
<th>Case</th>
<th>((LR)_{yz})</th>
<th>((RL)_{yz})</th>
<th>((A^*)_{yz})</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = z \in \mathcal{P}(x) )</td>
<td>( b_i^c )</td>
<td>( c_i^c )</td>
<td>( d - 2c_i^c )</td>
</tr>
<tr>
<td>( 2(y, z) = 2 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( 2(y, z) = 2(x, z) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>otherwise</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Result follows.

(iii): (iii) Similar
LEM 31

(i) \[ R = \frac{AA^* - A^*A + 2A}{4} \]

(ii) \[ L = \frac{A^*A - AA^* + 2A}{4} \]

pf Eliminate \( A \) using \( A = R + L \) and simplify using \( L30 \). \( \Box \)
Prop 3.2: We have

\[(i) \quad A^2 A^* - 2AA^* A + A^* A^2 = 4A^* \]

\[(ii) \quad A^* A^2 - 2A^* AA^* + AA^* A^2 = 4A \]

Proof (i):

\[
LHS = \left[ A_1 \left[ \begin{array}{c} A, A^* \end{array} \right] \right] \\
= \left[ A_1 \left[ \begin{array}{c} R+L, A^* \end{array} \right] \right] \\
= 2 \left[ \begin{array}{cc} R+L, & R-L \end{array} \right] \\
= 2 \left[ \begin{array}{c} L, R \end{array} \right] - 2 \left[ \begin{array}{c} R, L \end{array} \right] \\
= 4A^* \]

(ii) Similar.
Recall the primitive idempotent $E_i$ from $\theta_i = 0 - 2i \quad (0 \leq i \leq 0)$.

Prop 33 $A^k$ is a dual adjacency matrix with respect to and $E_i; i = 0$. Moreover

$$E_i A^k E_i = 0 \quad (0 \leq i \leq 0)$$

pf We show

$$E_i A^k E_i = 0 \text{ if } |i-j| \neq 1 \quad (0 \leq i, j \leq 0)$$

Let $i, j$ be given. By Prop 32 (c)

$$0 = E_i \left( A^2 A^k - 2AA^kA + A^k A^2 - 4A^k \right) E_j$$

$$= E_i A^k E_j \left( \frac{\theta^2 - 2\theta_i \theta_j + \theta_j^2 - 4}{(\theta_i - \theta_j)^2 - 4} \right)$$

$$= E_i A^k E_j \left( \frac{\theta_i - \theta_j}{(\theta_i - \theta_j - 2)(\theta_i - \theta_j + 2)} \right)$$

$$= 4 \left( i - j + 1 \right) \left( i - j - 1 \right)$$

So

$$E_i A^k E_i = 0 \text{ if } |i-j| \neq 1$$
Earlier we defined the raising matrix $R$ and lowering matrix $L$.

We now define dual versions $R^*$ and $L^*$

**DEF 34** For $Q_0$ define

$$R^* = \sum_{i=0}^{D} E_i a_i A^* E_i$$

$$L^* = \sum_{i=0}^{D} E_i a_i A^* E_i$$

We observe

$$\overline{R^*} = R^*, \quad \overline{L^*} = L^*, \quad (R^*)^T = L^*$$

$$A^* = R^* + L^*$$

$$R^* E_i V \leq E_i a_i V \quad \text{if} \quad i \leq 0$$

$$L^* E_i V \leq E_i a_i V \quad \text{if} \quad i \leq 0$$
LEM 35. We have

(i) \( R^* = \frac{A^x A - AA^x + 2A^x}{4} \)

(ii) \( L^* = \frac{AA^x - A^x A + 2A^x}{4} \)

\( \rho(f, c) \) write

\( C = \frac{A^x A - AA^x + 2A^x}{4} \)

View

\( C = I C I \)

\( I = \sum_{i=0}^{\infty} E_i \)

\( = \sum_{0 \leq i \leq 0} E_i C E_i \)

\( = \sum_{0 \leq i \leq 0} E_i A^x E_i \\frac{\theta_i - \theta_i + 2}{4} \)

\( = R^* \)

(6.61) Similar
LEM 36. We have

(i) \( L^* R^* - R^* L^* = A \)

(ii) \( A L^* - L^* A = 2L^* \)

(iii) \( A R^* - R^* A = -2R^* \)

pf. To verify each equation, eliminate \( L^* R^* \) using L35 and evaluate the result using Prop 32.

\( \square \)
Recall the Lie algebra \( \mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C}) \) is the \( \mathbb{C} \)-vector space consisting of the \( 2 \times 2 \) matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) that have trace 0, together with the Lie bracket

\[
[ u, v ] = uv - vu \quad u, v \in \mathfrak{sl}_2
\]

\( \mathfrak{sl}_2 \) has a basis

\[
e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

and

\[
[e, f] = h
\]

\[
[h, e] = 2e
\]

\[
[h, f] = -2f
\]
DEF 37  The universal enveloping algebra \( U(\mathfrak{sl}_2) \) is the (associative)
\( C \)-algebra with 1 defined by generators
\( E, F, H \) and relations
\[
EF - FE = H \\
HE - EH = 2E \\
HF - FH = -2F
\]

THM 38. For \( Q_0 \in \mathfrak{sl}_2 \) surjective \( C \)-alg

homomorphism
\[
U(\mathfrak{sl}_2) \rightarrow T
\]

that sends
\[
E \rightarrow L \\
F \rightarrow R \\
H \rightarrow A^*
\]
By Thm 38 the standard module $V$ becomes an $sl_2$-module such that

<table>
<thead>
<tr>
<th>generator</th>
<th>$e$</th>
<th>$f$</th>
<th>$h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>action on $V$</td>
<td>$L$</td>
<td>$R$</td>
<td>$A^*$</td>
</tr>
</tbody>
</table>

---

Define

$$a = e + f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$a^* = h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

On the $sl_2$-module $V$

<table>
<thead>
<tr>
<th>generator</th>
<th>$a$</th>
<th>$a^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>action on $V$</td>
<td>$A$</td>
<td>$A^*$</td>
</tr>
</tbody>
</table>

One checks

$$\left[ a, [a, a^*] \right] = 4a^*$$  \hspace{1cm} (compare with Prop 32)

$$\left[ a^*, [a^*, a] \right] = 4a$$