Ch 9

Until further notice,

$R$ is a commutative ring with $1 \neq 0$

Let $x =$ indeterminate

$R[x] =$ ring of polynomials in $x$ with coeffs in $R$

View $R$ is a subring of $R[x]$

For $0 \neq f \in R[x]$ write

\[ f = a_0 + a_1 x + \ldots + a_n x^n \]

Declare $\deg (0) = 0$
LEM 1  TFAE

(i) $R$ is an integral domain

(ii) $R[x]$ is an integral domain

pf 

(i) $\rightarrow$ (ii) Given $n \in R$ and $f, g \in R[x]$

Show $fg \neq 0$

Write

\[
f = a_0 + a_1 x + \cdots + a_r x^r \quad a_i \in R
\]

\[
g = b_0 + b_1 x + \cdots + b_s x^s \quad b_j \in R
\]

\[
fg = \left( \sum_{i=0}^{r} a_i x^i \right) \left( \sum_{j=0}^{s} b_j x^j \right)
\]

\[
= \sum_{i=0}^{r} \sum_{j=0}^{s} a_i b_j x^{i+j}
\]

\[
= a_r b_s x^{r+s} + \text{lower terms}
\]

\[
a_r b_s \neq 0 \quad \text{since} \quad R \text{ is int domain}
\]

So

\[
fg \neq 0
\]

(ii) $\rightarrow$ (i) Since $R$ is subring of $R[x]$
LEM 2 For an ideal \( I \) of \( R \) the following sets are equal:

\[ J_1 \]

(i) the ideal of \( R[x] \) generated by \( I \)

(ii) the polynomials in \( x \) that have all coefficients in \( I \)

[We call this common set \( I[x] \).]

pf \( J_1 \subseteq J_2 \):

Given \( g \in J_1 \),

write 

\[ g = f_1 a_1 + f_2 a_2 + \ldots + f_r a_r \]

\( f_i \in R[x], \ a_i \in I \)

Show each term \( f_i a_i \) is in \( J_2 \)

For \( f \in R[x] \) and \( a \in I \)

show \( fa \in J_2 \)

Write 

\[ f = b_0 + b_1 x + \ldots + b_a x^a \]

\( b_i \in R, \ a > 0 \)

So 

\[ fa = \frac{b_0 a_1}{I} + \frac{b_1 a_1}{I} x + \ldots + \frac{b_a a_1}{I} x^a \in J_2 \]

\( J_2 \subseteq J_1 \):

For \( g \in J_2 \) write 

\[ g = a_0 + a_1 x + \ldots + a_b x^b \]

\( a_i \in I, \ b > 0 \)

\[ g = \sum_{i=0}^{b} a_i x^i \quad \in \quad J_1 \]

\[ \square \]
Given ring homomorphism:

\[ \phi : R \rightarrow S \]

then another ring homomorphism:

\[ \overline{\phi} : \mathbb{R}[x] \rightarrow \mathbb{S}[x] \]

\[ \sum_{i} a_i x^i \mapsto \sum_{i} \phi(a_i) x^i \]

- \( \phi \) is surjective iff \( \overline{\phi} \) is surjective.

- Let \( I = \ker(\phi) \). Then

\[ \ker(\overline{\phi}) = I[x] \]
Consider the quotient module 

\[ \mathcal{Q} \mathcal{I} \]

Write \( S = R/I \)

Let \( M \) be the following rings and isomorphic to

\[ R[x]/(x) \]

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Now \( \hat{\phi} : R[x] \to S[x] \) is surj with ker \( I[x] \).

Now \( \hat{\phi} \) induces ring iso

\[
\begin{array}{c}
R[x]/I[x] \\
\downarrow \\
\hat{\phi}(f)
\end{array}
\]

Result follows.
\textbf{LEM 4} Given an ideal } I \subset R \\
Consider ideal } I[x] \subset R[x]. \quad \text{TFAE}

(i) \quad I \text{ is prime}

(ii) \quad I[x] \text{ is prime.}

\text{pf} \\
I \text{ prime} \\
\implies R/I \text{ is integral domain} \quad \text{by LEM 1}

\implies R/I[x] \text{ is int. domain} \quad \text{by LEM 3}

\implies R[x]/I[x] \text{ is int domain.}

\iff I[x] \text{ is prime.} \\
\square
Given mutually commuting indeterminates

\[ x_1, x_2, \ldots, x_n \]

Define

\[ R[x_1, x_2, \ldots, x_n] = \text{ring of polynomials in } x_1, x_2, \ldots, x_n \]

that have all coefficients in \( R \).

View

\[ R[x_1, x_2, \ldots, x_n] = R[x_1, \ldots, x_{n-1}][x_n] \]

A monomial in \( R[x_1, \ldots, x_n] \) has form

\[ x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n} \]

This monomial has degree with respect to \( (e_1, e_2, \ldots, e_n) \)

and total degree \( e_1 + e_2 + \cdots + e_n \)

(overall degree)
Each $f \in \mathbb{R}[x_1, \ldots, x_n]$ is a finite sum

$$f = \sum_{i=1}^{l} a_i f_i \quad l \geq 0$$

where

$$a_i, f_i \in \mathbb{R} \quad (i \leq l),$$

$f_i, f_2, \ldots, f_l$ are not all zero monomials.

Define

$$\text{deg}(f) = \max \left\{ \text{deg}(f_i) \mid 1 \leq i \leq l \right\}$$

Call $f$ **homogeneous of degree** $k$ whenever

$$\text{deg}(f_i) = k \quad 1 \leq i \leq l$$

Each element of $\mathbb{R}[x_1, \ldots, x_n]$ is a sum of homogeneous elements.

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Given a field $F$

We have seen $F[x]$ is Euclidean domain.

We now consider this in more detail.
LEM 5 Given field $F$

Given $a, b \in F[x]$ with $b \neq 0$

Then there exist unique pair $q, r \in F[x]$ such that

$$a = bq + r$$

and

$$r = 0 \quad \text{or} \quad \deg(r) < \deg(b)$$

pf

Show $q, r$ exist.

Call the ordered pair $a, b$ a counterexample whenever $q, r$ do not exist.

Assume $a, b$ is a counterexample.

**Obs**

$$\deg(a) \geq \deg(b)$$

else

satisfied by

$$r = a, \quad q = 0$$

Among all counterexamples, WLOG assume

$$\deg(a) - \deg(b)$$ is minimal
Write
\[ a = a_0 + a_1 x + \ldots + a_r x^r \quad a_i \in F \]
\[ b = b_0 + b_1 x + \ldots + b_s x^s \quad b_j \in F \]
\[ 0 \leq a \leq r \]
\[ b_a \neq 0 \Rightarrow b_a^{-1} \text{ exists in } F \]

Define
\[ A = a - b_a^{-1} a x^{-a} \in F[x] \]

By construction, since \( A \) coef + \( x^r \) is 0
\( A \) does not involve \( x^r, x^{r+1}, x^{r+2}, \ldots \)

So
\[ A = 0 \quad \text{deg}(A) < r = \text{deg}(a) \]

So
\[ A, b \] is not a counterexample.

So
\[ \exists Q, r \in F[x] \text{ s.t.} \]
\[ A = bQ + r \]

And
\[ r = 0 \quad \text{deg}(r) < \text{deg}(b) \]
We have

\[ A = b \cdot q + r \]

\[ a = b \cdot r_{a} \cdot b_{a} \cdot x^{-a} \]

\[ q = b \left( \frac{q + a \cdot b_{a} \cdot x^{-a}}{1} \right) + r \]

\[ \text{call this } q' \]

Now \( q, r \) satisfy \( x, x \).  

So \( a, b \) not a counterexample, etc.

We have shown \( q, r \) exists.

Next we show \( q, r \) is unique.

Suppose \( \exists q', r' \in F[x] \) s.t.

\[ a = bq' + r' \]

\[ r' = 0 \quad \text{a} \quad \deg(r') < \deg(b) \]

Show \( q = q' \) and \( r = r' \). Assume \( q \neq q' \) else done.

Combine \( x, x \) to get

\[ b(q - q') = r' - r \]
So
\[
\deg(b) + \deg(q - q') = \deg(r - r')
\]

we have
\[
\deg(b) + \deg(q - q') = \deg(r - r') \leq \deg(b)
\]

Note.

So
\[
q = q' \quad \text{and} \quad r = r'
\]