Recall $R$ is a commutative ring with $1 
eq 0$.

In $R$-modules, we continue to discuss rank and linear independence.

Assume $R$ is an integral domain.

Given an $R$-module $V$ and elements $\{v_i\}_{i=1}^n$ in $V$, consider the $R$-submodule

$$W = \sum_{i=1}^n Rv_i$$

Consider the map

$$\psi : R^n \rightarrow W$$

$$(c_1, c_2, \ldots, c_n) \rightarrow c_1v_1 + c_2v_2 + \cdots + c_nv_n$$

Observe $\psi$ is a surjective $R$-module homomorphism.

Moreover TFAE

(i) $\{v_i\}_{i=1}^n$ are linearly independent.

(ii) $\psi$ is an $R$-module isomorphism.
LEM 14. Assume $R$ is an integral domain.

Given an $R$-module $V$ and $R$-submodules $U, U'$ of $V$ such that $U \cap U' = 0$.

Given $m$ indep elements of $U$:

$u_1, u_2, \ldots, u_n$

Given $m$ indep elements of $U'$:

$u'_1, u'_2, \ldots, u'_m$

Then

$u_1, u_2, \ldots, u_n, u'_1, u'_2, \ldots, u'_m$

are $m$ indep in $V$.

pf. Given \{$c_i^1\}_{i=1}^n$, \{$c_i'^1\}_{i=1}^m \in R$

\[ \sum_{i=1}^n c_i^1 u_i + \sum_{i=1}^m c_i'^1 u'_i = 0 \]

\[ \forall u \in U, \forall u' \in U' \]

\[ u \in U \cap U' = 0 \]

\{c_i, i=1, \ldots, n\} all 0 since \* lin indep

\{c_i'^1, i=1, \ldots, m\} all 0 since \* lin indep

\[ \square \]
LEM 15 Assume \( R \) is an integral domain

Given an \( R \)-module \( V \) and linearly independent elements

\[ u_1, u_2, \ldots, u_n \in V \]

If \( A \in \text{Mat}_n(R) \) consider the elements

\[ v_j = \sum_{i=1}^n A_{ij} u_i \quad \text{for} \quad j = 1, \ldots, n \]

then \( v_j \) are linearly independent if \( \det(A) \neq 0 \)

pf. \( v_j \) are linearly independent if \( \det(A) \neq 0 \)

\[ c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \in R^n \]

we have

\[ \sum_{j=1}^n c_j v_j = \sum_{j=1}^n c_j \left( \sum_{i=1}^n A_{ij} u_i \right) = \sum_{i=1}^n \left( \sum_{j=1}^n A_{ij} c_j \right) u_i \]

so

\[ \sum_{j=1}^n c_j v_j = 0 \quad \text{if} \quad A c = 0 \]

Recall \( A c = 0 \) has a non-zero solution \( c \) if \( \det(A) \neq 0 \)

Result follows.
Cor 16. Assume $R$ is an integral domain.

Given an $R$-module $V$ and lin indep elements

$$a_1, a_2, \ldots, a_n \in V$$

Given

$$0 \neq a_i \in R$$

then the elements

$$a_1v, a_2v, \ldots, a_nv$$

are lin indep.

pf. The matrix

$$A = \text{diag} (a_1, a_2, \ldots, a_n)$$

has det $a_1 a_2 \cdots a_n$.

$\neq 0$ since $R$ is integral domain.

Result follows by Lem 15.

$\square$
Assume $R$ is an integral domain.

Given an $R$-module $V$

Given lin. ind. elements

$$v_1, v_2, \ldots, v_n \in V \quad n = \text{rank}(V)$$

So the $R$-submodule

$$W = \sum_{i=1}^{n} Rv_i$$

is iso $R^n$

LEM 17 With above notation, the quotient $R$-module $V/W$ is torsion.

pf $\forall v \in V$ the elements $v, v_1, v_2, \ldots, v_n$ are lin. ind.

So if $a, a_1, a_2, \ldots, a_n \in R$ not all 0

St

$$av + \sum_{i=1}^{n} a_i v_i = 0$$

$a \neq 0$ since $v, v_1, v_2, \ldots, v_n$ are lin. ind. Also

$$av \in \sum_{i=1}^{n} Rv_i = W$$

So in $V/W$

$$a(v+W) = av + W = W$$

So $v+W$ is torsion.
We now reverse the logical direction.

**Lemma 18** Assume \( R \) is an integral domain. Given an \( R \)-module \( V \) and \( n \geq 0 \), assume \( \exists \) \( R \)-submodule \( W \) of \( V \) that is isomorphic to \( R^n \) and \( V/W \) is torsion. Then

\[
\text{rank} (V) = n \leq 1/n
\]

**Proof**

Observe

\[
n = \text{rank} (W) \leq \text{rank} (V) = N
\]

**Show** \( n \geq N \):

\[
\exists \text{ linearly independent } v_1, v_2, \ldots, v_n \in V
\]

For all \( n \leq N \) \( \exists \) \( \alpha_1, \alpha_2, \ldots, \alpha_N \in R \) such that

\[
\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_N v_N \in W
\]

Since \( V/W \) is torsion

By Cor 16

\[
a, b, a_2 v_2, \ldots, a_N v_N \text{ linearly dependent}
\]

Now

\[
n \geq N \text{ since } W \text{ has rank } n. \quad \square
\]
Prop 19  Assume \( R \) is an integral domain.

Given \( R \)-modules \( U, V \).

Then for the direct product \( R \)-module \( U \times V \),

\[
\text{rank}(U \times V) = \text{rank}(U) + \text{rank}(V)
\]

\[
\begin{array}{c|c|c}
& N & m \times n \\
\hline
\end{array}
\]

pf Write \( W = U \times V \)

View \( U, V \) as \( R \)-submodules of \( W \), so

\[
W = U + V \quad (\text{dir. sum})
\]

\[
\exists \ \text{lin. indep} \\
\quad u_1, u_2, \ldots, u_m \in U
\]

\[
\exists \ \text{lin. indep} \\
\quad v_1, v_2, \ldots, v_n \in V
\]

Show \( N \geq mn \)  

The sum \( U + V \) is direct, so

by LEM 14,

\[
\begin{array}{c|c|c}
& u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n & \text{lin. indep in } W \\
\hline
\end{array}
\]
Show \( N \leq \min m \)

\[ \exists \text{ lin indep elements} \]

\[ w_1, w_2, \ldots, w_N \in W \]

For \( 1 \leq i \leq N \) write

\[ w_i = w_i^+ + w_i^- \]

\[ U \quad V \]

\[ U \quad V \]

Obs

\[ w_i^+, u_i, u_2, \ldots, u_m \in \text{ lin dep} \]

So

\[ \exists \ a \in \mathbb{R} \text{ st} \]

\[ a_i w_i^+ \in \left( R_{u_1} + R_{u_2} + \cdots + R_{u_m} \right) \]

\[ \overset{U}{\quad V} \]

Obs

\[ w_i^-, v_i, v_2, \ldots, v_n \in \text{ lin dep} \]

So

\[ \exists \ a \in \mathbb{R} \text{ st} \]

\[ b_i w_i^- \in \left( R_{v_1} + R_{v_2} + \cdots + R_{v_n} \right) \]

\[ \overset{V}{\quad} \]
\[ \text{Obs} \]
\[ a_i b_i w_i = a_i b_i (w_i^+ + w_i^-) \]
\[ = b_i (a_i w_i^+) + a_i (b_i w_i^-) \]
\[ \subseteq \bar{U} \oplus \bar{V} \]

By Cor 16
\[ a_i b_i w_i \in \mathbb{N} \]
are lin indep.

By above lemma we have \( R \)-module isomorphisms
\[ \bar{U} \cong \mathbb{R}^m, \quad \bar{V} \cong \mathbb{R}^n \]

So \( \bar{U} + \bar{V} \cong \mathbb{R}^{m+n} \)

So \( \text{rank} (\bar{U} + \bar{V}) = mn \)

But \( \bar{x} \) gives \( N \) lin indep elements in \( \bar{U} + \bar{V} \).

So \( N \leq mn \)
Assume $R$ is integral domain.

Given $f,g \in R$-module $V$,

**Describe $V$**

Write $V = \sum_{i=1}^{n} Rv_i$.

Recall the map

\[ \psi: \mathbb{R}^n \rightarrow V \]

\[ \left( a_1, a_2, \ldots, a_n \right) \rightarrow a_1v_1 + a_2v_2 + \ldots + a_nv_n \]

is surjective $R$-module homomorphism.

Let $W = \ker \psi$.

So $W$ is $R$-submodule of $\mathbb{R}^n$.

$\psi$ induces $R$-module $\psi$.

\[ \mathbb{R}^n/W \rightarrow V \]

\[ x + W \rightarrow \psi(x) \]

To describe the solutions of $V$, it suffices to describe the $R$-submodule $W$ of $\mathbb{R}^n$. 

Given $R$-submodule $W$ of $R^n$.

By LEM 13, $W$ is torsion-free.

Write $m = \text{rank}(W)$.

Obv $m \leq \text{rank}(R^n) = n$.

Natural question: does there exist an $R$-module $W$ so $W = R^m$?

We will show: ans is "No" in general.

ans is "yes" if $R$ is a PID.

The next example illustrates why the ans is No in general.