Assume $R$ is a PID

Our goal: For fg $R$-modules, show

the uniqueness of the invariant factors

and elementary divisors

let $U$ denote a fg $R$-module of rank $t$

We saw $\exists R$-submodule $F$ of $U$ stab both

(i) the sum $U = \text{Tor}(U, F)$ is direct

(ii) $\exists R$-module iso $F \cong R^t$

the inv factors and elem divisors of $U$ both describe

how to decompose $\text{Tor}(U)$ as a direct sum of cyclic

$R$-modules. To show the inv factors and elem divisors

are unique, WLOG $U$ is torsion.
Assume $U$ is torsion and non-zero.

Write

$$\text{Ann}(U) = \mathbb{R} \phi_i$$

other $\mathbb{R}$

Factor $D$ into irreducibles:

up to associate

$$p = \phi_1 \phi_2 \cdots \phi_r$$

$r, e_1, e_2, \ldots, e_r > 0$

$p_i \phi_j$ not associated if $i \neq j$ (i.e. $i \neq j$)

For $1 \leq i \leq r$ define

$$U(i) = \left\{ v \in U \mid \exists n \geq 1 \text{ s.t. } \phi_i^n v = 0 \right\}$$

One checks $U(i)$ is an $\mathbb{R}$-submodule of $U$

Call $U(i)$ the $\phi_i$-primary component of $U$

One checks

$$\text{Ann}(U(i)) = \mathbb{R} \phi_i^e$$

By $\text{th. 27}$ (or directly from $D_\sigma \in \text{KM}$),

$$U = U(1) + U(2) + \cdots + U(r)$$

discriminant
LEM 28  Given a prime $p \in R$ and an $R$-module $R/R_p$, it is a field $\text{F}$.

Given an $R$-module $V$.

(i) Assume $V = R^t$ is free. Then

$\exists$ $R$-module iso

$V/\rho V = \text{F}^t$

(ii) Assume $V = R/Ra$ for $a \in R$.

Then $\exists$ $R$-module iso

$V/\rho V = \begin{cases} \text{F} & \text{if } \rho/a \\ 0 & \text{if } \rho/a \end{cases}$

(iii) Assume

$V = R/Ra_1 \times R/Ra_2 \times \cdots \times R/Ra_k$

with

$\rho/a_i \neq 0_{Ra_i}$

Then $\exists$ $R$-module iso

$V/\rho V = \text{F}_k$
pf (i) the map

\[ \psi : V \rightarrow \mathbb{F}^t \]

\[(x_1, x_2, \ldots, x_t) \mapsto (x_1 + x_2 + \cdots + x_t) \]

is surjective \( R \)-module hom.

OBS

\[ \ker(\psi) = pV \]

So \( \psi \) induces \( R \)-module iso \( V/pV \cong \mathbb{F}^t \).

(ii) Write \( W = V/pW \)

Suppose \( p \neq 0 \)

By construction \( pW = 0 \) and \( 1W = 0 \)

We have \( \gcd(p, a) = 1 \), so \( 1W = 0 \)

So \( V/pW = 0 \)

Suppose \( p = 0 \)

So \( Rq \leq Rp \)
\[
\text{E} \
\begin{align*}
\text{sur} & \quad \text{R-module hom} \\
& \quad \text{h} \\
& \quad V \\
\phi & \quad R/R_n \rightarrow R/R_p \\
(r + R_n) & \rightarrow r + R_p \\
\text{We have} & \quad \ker(\phi) = pV \\
\text{So} & \quad \phi \text{ induces } R\text{-mod iso} \\
V/pV & \cong IF
\end{align*}
\]

(iii) Use (ii)
LEM 29  Given a prime \( p \in \mathbb{P} \).

Given positive integers

\[ d_1 \leq d_2 \leq \ldots \leq d_a \quad 0 \leq a, b < \infty \]

\[ \beta_1 \leq \beta_2 \leq \ldots \leq \beta_t \]

TFAE

(i) These \( \mathbb{R} \)-modules are isomorphic:

\[
\mathbb{R}/\mathbb{R} p^{d_1} \times \mathbb{R}/\mathbb{R} p^{d_2} \times \cdots \times \mathbb{R}/\mathbb{R} p^{d_a}
\]

\( V = \mathbb{R}/\mathbb{R} p^{\beta_1} \times \mathbb{R}/\mathbb{R} p^{\beta_2} \times \cdots \times \mathbb{R}/\mathbb{R} p^{\beta_t} \)

W = \mathbb{R}/\mathbb{R} p^{a} \times \mathbb{R}/\mathbb{R} p^{b} \times \ldots \times \mathbb{R}/\mathbb{R} p^{a}

(ii) \( a = b \) and

\[ d_i = \beta_i \quad 1 \leq s \leq a \]

\( \rho f \) (i) \( \Rightarrow \) (ii)

\[ \text{Ann} (V) = \mathbb{R} p^{d_a} \]

\[ \text{Ann} (W) = \mathbb{R} p^{\beta_t} \]

So

\[ d_a = \beta_t \]

We proceed by induction on \( \star \)

Case \( \star = 1 \)

Here

\[ d_1 = 1 \quad 1 \leq i \leq a \]

\[ \beta_1 = 1 \quad 1 \leq i \leq t \]
\[ S_0 \quad V = F^A, \quad W = F^2 \]
\[ F = R/R_p \]

Viewing \( V, W \) as free \( F \)-modules (ie vs over \( F \)), we see \( A = \mathbb{C} \).

\[ \text{Case } \quad x = x_1 \]
\[ \text{We have } \quad R\text{-module} \]
\[ pV = pW \]

\[ \text{Obs } \quad pV = \]
\[ \frac{R}{R_p \times \cdots \times R} \quad \frac{R}{R_p \times \cdots \times R} \quad \frac{R}{R_p \times \cdots \times R} \]

\[ \text{If } x_1 = 1 \]
\[ \text{Also } \quad pW = \]
\[ \frac{R}{R_p \times \cdots \times R} \quad \frac{R}{R_p \times \cdots \times R} \quad \frac{R}{R_p \times \cdots \times R} \]
Consider the sequences

\[ a_{i-1} \leq a_{i-1} \leq \ldots \leq a_n \]
\[ b_{i-1} \leq b_{i-1} \leq \ldots \leq b_n \]

By induction,

sequence of non-zero terms in \( \mathbb{F} \)

sequence of non-zero terms in \( \mathbb{F} \)

Also \( a = t \) since by Lemma 28,

\[ V/_{p} V \Rightarrow W/_{p} W \]

Now their sequence coincide:

\[ a_i \leq x_i \leq \ldots \leq a_n \]
\[ b_i \leq x_i \leq \ldots \leq b_n \]

(iii) \( \Rightarrow \) (iii) clear.
Given $f : R$-module $U \to V$

TFAE

(i) $f$ an $R$-module iso $U \cong V$

(ii) $U$, $V$ have the same free-rank and same
list of elem divisors
(up to perm + assoc)

pf (i) $\Rightarrow$ (ii): We saw free-ranks = ranks, so these
are same.

We have $\text{Tor}(U) \cong \text{Tor}(V)$

WLOG $U = \text{Tor}(U)$, $V = \text{Tor}(V)$

Recall

$U = \text{ds of its primary components}$

$V = \ldots$

So for each prime $p \in R$

$p$-primary comp of $U \cong p$-primary comp of $V$

WLOG $f$ prime $p \in R$ s.t.

$U = \text{its } p$-primary comp

$V = \ldots$

Now $U$, $V$ have same elem divisors by LEM 29.

(iii) $\Rightarrow$ (i) $\checkmark$
Thm 31: Given by R-modules U, V

TFAE

(i) \exists R-module \alpha \text{ such that } U = V

(ii) U, V have same join rank and same list of invariant factors (up to assoc.)

pf: the divisors and then divisors determine each other. Result follows by Thm 30.