LECTURE 39  Wednesday  April 27

The Rational Canonical Form

Let $F$ denote a field.

For an indeterminate $x$, recall the polynomial ring $F[x]$.

We saw the ring $F[x]$ is a P.I.D.

Let $V$ denote an $F[x]$-module.

Then $V$ is an $F$-module (as a vector space over $F$).

And the map

$$T : V \rightarrow V$$

$$v \rightarrow xv$$

is an $F$-linear transformation.

Assume the $F[x]$-module $V$ is $f.g.$.

Recall that $V$ is the direct sum of finitely many cyclic $F[x]$-submodules.
Assume the $F[x]$-module $V$ is cyclic and nonzero.

So if $0 \neq v \in V$ s.t.

$$V = F[T]v$$

So the vector space $V$ is spanned by

$$v, \quad Tv, \quad T^2v, \quad \ldots$$

Case: the $F[x]$-module $V$ is free.

The vectors $v$ form a basis for $V$.

The $\dim V$ is finite.

Case: the $F[x]$-module $V$ is torsion.

Write

$$\text{Ann}(V) = F[x]m(x)$$

where $m(x)$ is unique up to multi by nonzero scalar in $F$.

We call $m(x)$ is monic.

Obs.

$m(x)$ is the monic poly of least degree such that

$$m(T) = 0$$

Call $m(x)$ the \underline{minimal} polynomial for $T$. 

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Write
\[ m(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_k x^k + r x^k \]
k ≥ 1, \quad b_0, b_1, \ldots, b_k, r \in F

Then the vectors
\[ v, T v, T^2 v, \ldots, T^k v \]
form a basis from \( v \) vs \( V \) and
\[ T^k v = -b_0 v - b_1 T v - b_2 T^2 v - \ldots - b_k T^k v \]
So \( v \) vs \( V \) has dim \( k \)

Let \( A \) = matrix in \( \text{Mat}_k(F) \) that represents \( T \)
with basis \( \{ x^k \} \)

Then
\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & -b_0 \\
0 & 0 & \cdots & 0 & -b_1 \\
0 & 0 & \cdots & 0 & -b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & -b_k
\end{pmatrix}
\]

Call \( A \) the companion matrix for the poly \( m(x) \)
LEM 1  Given an $F[x]$-module $V$

TPAE

(i) $F[x]$-module $V$ is finitely generated

(ii) Vector space $V$ is finite-dimensional $\dim |V| < \infty$

pf  (i) $\rightarrow$ (ii) $V$ is direct sum of finitely many cyclic $F[x]$-submodules, each of which is torsion and hence finitely generated.

(ii) $\rightarrow$ (i) Write $n = \dim |V|$

Pick a basis $\{v_1, \ldots, v_n\} \rightarrow V$

$V = \sum_{i=1}^{n} Fv_i$

$= \sum_{i=1}^{n} F[x]v_i$

So $F[x]$-module $V$ is finitely generated.

Now $F[x]$-module $V$ is direct sum of cyclic $F[x]$-submodules. Now one free so all are torsion.

So $F[x]$-module $V$ has rank 0 so $V$ is torsion. $\Box$
Until further notice the \( F[x] \)-module \( V \) satisfies

**LEMMA** (i), (ii)

Consider the invariant factor decomp of the \( F[x] \)-module \( V \).

The inv factors are (monic polynomials in \( F[x] \)):

\[
m_1(x), m_2(x), \ldots, m_r(x)
\]

each with degree \( \geq 1 \) and

\[
m_1(x) \mid m_2(x) \mid \cdots \mid m_r(x)
\]

Recall

\[
\text{Ann}(V) = F[x] m_r(x)
\]

\( m_r(x) \) is monic pol at least deg \( \geq 5 \) \( m_r(T) = 0 \)

\[
\text{min poly of } T^n
\]
Thm 2: With above notation,

exists basis \( f_a \) vs \( V \) with respect to which the matrix representing \( T \) is

\[
\begin{pmatrix}
C_1 & & \\
& C_2 & \\
& & \ddots
\end{pmatrix}
\]

"Rational Canonical Form"

where \( C_i \) is the companion matrix of \( m_i(x) \) \( \forall i \leq \sigma \).

pf: F[x]-module \( V \) is ds of cyclic F[x]-submodule

\[
V = F[x]v_1 + F[x]v_2 + ... + F[x]v_r
\]

with

\[
\text{Ann} \left( F[x]v_i \right) = F[x] m_i(x) \quad \forall i \leq \sigma
\]

\( F[x]v_i \) vs \( F[x]v_i \) has basis w.r.t which main

rep \( T = C_i \). Result follows. \( \square \)
DE: \( \exists F \) \, \lambda \in F \) define
\[ V_\lambda = \left\{ v \in V \mid T(v) = \lambda v \right\} \]

Call \( \lambda \) an eigenvalue of \( T \) iff \( V_\lambda \neq \emptyset \)

In this case call \( V_\lambda \) the \( \lambda \)-eigenspace for \( T \)

LEM: \[ \text{Pick a basis } \{ v_1, v_2, \ldots, v_n \} \]

\[ \text{Let } B \text{ denote the matrix rep } T \text{ w.r.t. } \{ v_1, v_2, \ldots, v_n \} \]

Then \( \det(B) \) is indep \( \lambda \).

pf: \[ \text{Pick a second basis } \{ w_1, w_2, \ldots, w_n \} \]

Let \( S \in \text{Mat}_{n,n}(F) \) denote the transition matrix
\[ \text{from } \{ v_1, v_2, \ldots, v_n \} \text{ to } \{ w_1, w_2, \ldots, w_n \} \]
The matrix rep. \( T \) w.r.t. \( \{ w_1, w_2, \ldots, w_n \} \) is
\[ S^{-1}BS \]

Now \[ \det(\lambda^{-1}BS) = \det(S) \det(B) \det(S^{-1}) \]
\[ = \det(B) \lambda \]
\[ \square \]
DEF 5  By the determinant of $T$, we mean $\det(B)$ where the matrix $B$ represents $T$ with some basis for $V$.

Prop 6  For $\lambda \in F$ $T \in M_n(F)$

(i) $\lambda$ is an eigenvalue of $T$

(ii) $\lambda I - T$ is not invertible

(iii) $\det(\lambda I - T) = 0$

pf  elem. lin. alg. $\Box$
Def: The characteristic polynomial of $T$ is the following poly in $F[x]$: 

$$ \det (xI - T) $$

$C(x) = \det (xI - T)$ is a monic with degree $= \dim (V)$.

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We now describe how the char poly $C(x)$ is related to the $m_p(x)$. 
Prop 8  

With above notation

\[ C(x) = m_1(x) m_2(x) \ldots m_r(x) \]

pf  

Consider the basis \( \mathbf{v} \) as in \( \text{Ann} Z \).

Rel this basis

\[ \mathbf{T} = \begin{pmatrix} \mathbf{c} & 0 \\ \mathbf{c} & 0 \\ \mathbf{c} & 0 \\ \mathbf{c} & 0 \end{pmatrix} \]

so

\[ xI - T = \begin{pmatrix} xI - c_5 \\ xI - c_4 \\ xI - c_3 \\ xI - c_2 \end{pmatrix} \]

\[ \det (xI - T) = \prod_{i=1}^{r} \det (xI - c_i) \]

\[ \mathbf{m}_i(x) \]

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Cor 9. the min poly of T divides the char poly of T.

\[ m(x) \mid c(x) \]

\[ m(x) = m_r(x) \]

By Prop 8, \( m(x) \mid c(x) \)

\[ \square \]

Cor 10 (Cayley–Hamilton law)

\[ c(T) = 0 \]

where \( c(x) \) is the char poly of \( T \).

pf \[ c(T) = m(T) \cdot m_r(T) - \frac{m_r(T)}{1} \]

\[ = 0 \]
Corollary 11

The characteristic polynomial of $T$ divides some power of the minimal polynomial of $T$.

$$ \chi(x) \mid m^r(x) $$

Proof

By Proposition 8

$$ \chi(x) = m_1(x) m_2(x) \cdots m_r(x) $$

For (surjective)

$$ m_i(x) \mid m(x) $$

Write

$$ m(x) = m_i(x) M_i(x), \quad M_i(x) \in F[x] $$

Observation

$$ m(x)^r = m_1(x) M_1(x) \cdots m_r(x) M_r(x) $$

$$ = m_1(x) \cdots m_r(x) M_1(x) \cdots M_r(x) \chi(x) $$

So

$$ \chi(x) \mid m(x)^r $$