Lecture 41  Monday  May 2

The Jordan Canonical Form

Let $F$ denote a field

Let $V$ denote a finite dimensional vector space over $F$

Given a linear trans $T: V \to V$

For an indeterminate $x$, recall polynomial ring $F[x] = R$

View $V$ as an $F[x]$-module with

$$xv = Tv$$

$F[x]$-module $V$ is torsion since dim $V < \infty$

$$\text{Ann}(V) = Rm(x)$$

$m(x) = \text{min} \text{poly of } T$

$m(x) | c(x)$

$c(x) = \text{char} \text{poly of } T$
Consider the elementary divisor decomp

do R-module $V$.

To keep things simple we always assume:

Each root of $c(x)$ is contained in $F$ (ie each eigenvalue of $T$ is contained in $F$)

By the elem. divisor decomp,

$V$ is a direct sum of finitely many cyclic

$R$-submodules, each with annihilator generated by

a prime power in $R$.

Given a summand

$$Rv$$

$Ann (Rv)$ is generated by a prime power in $R$, denoted

$$p^r$$

$r \geq 1, p$ prime (and mimic)
Recall \( p | c(x) \) so by \( x \),

\[
p = x - \lambda \\
\lambda \in F
\]

**LEM 1** With above notation,

the vector space \( R\nu \) has a basis

\[
u, \ (x-\lambda)\nu, \ (x-\lambda)^2\nu, \ \cdots, \ (x-\lambda)^r\nu
\]

**pf** Show \( \star \) spans \( R\nu \):

- \( R\nu \) is spanned by

\[
u, \ x\nu, \ x^2\nu, \ \cdots
\]

Change vars \( x \to x-\lambda \)

- \( R\nu \) is spanned by

\[
u, \ (x-\lambda)\nu, \ (x-\lambda)^2\nu, \ \cdots
\]

By csmke \( (x-\lambda)^r\nu = 0 \)

So \( \star \) spans \( R\nu \)
Check **A** in dep

Given \[ x_i \in F \quad (0 \leq i \leq n) \] s.t.

\[
0 = \sum_{i=0}^{n} x_i (x - \lambda)^i \nu
\]

Show \[ x_i \neq 0 \quad \text{for} \quad 0 \leq i \leq n \]

Define

\[
f(x) = \sum_{i=0}^{n} x_i (x - \lambda)^i
\]

\[
f(x) \nu = 0
\]

\[
f(x) \in R \quad f(x) \nu
\]

\[
f(x) \in \text{Ann}(Rv)\]

\[
f(x) \text{ is divisible by } (x - \lambda)^r
\]

\[
\deg \leq r
\]

\[
\deg \nu
\]

So \[ f(x) = 0 \quad \text{in} \quad R \]

\[
ex_i \neq 0 \quad \text{for} \quad 0 \leq i \leq n
\]
It is convenient to put $\mathbf{x}$ in reverse order.

**LEM 2** With above notation, with respect to the basis

\[(x-\lambda)^{\nu}, \ldots, (x-\lambda)^{2\nu}, (x-\lambda)^{\nu}, \nu\]

the matrix representing $T$ is

\[
T = \begin{pmatrix}
\lambda & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & \ddots & 1 & 0 \\
0 & 0 & \ddots & \ddots & \lambda \\
0 & 0 & \ddots & \ddots & \ddots \\
\end{pmatrix}
\]

**pf** define

\[\nu_i = (x-\lambda)^{-i}\nu \quad \text{as needed}\]
\[ (x - \lambda) \nu_i = \nu_i \quad (1 \leq i \leq r) \]
\[ (x - \lambda) \nu_0 = 0 \]

So,

\[ x \nu_i = \lambda \nu_i + \nu_i \quad (1 \leq i \leq r) \]
\[ x \nu_0 = \lambda \nu_0 \]

\( x \) acts on \( T \) so the result follows.

**DEF 3** Referring to LEM 2.

The matrix \( \Phi \) is called the \( r \times r \) Jordan block with eigenvalue \( \lambda \).

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Return to only \( \nu \neq 0 \).
Thm 4: The \( V \) has a basis \( T \) such that the matrix representing \( T \) is

\[
\begin{pmatrix}
J_1 & & \\
& J_2 & \\
& & \ddots \\
& & & \ddots \\
& & & & J_s
\end{pmatrix}
\]

where each of \( J_1, J_2, \ldots, J_s \) is a Jordan block.

pf: By LEM2, and since \( V \) is a direct sum of cyclic \( R \)-submodules.
Note 5

Referring to Thm 4, we can recover the elementary divisors of $T$ from its Jordan canonical form.

Given an $r 	imes r$ Jordan block $J$ with eigenvalue $\lambda$

\[ (x-\lambda)^r = \text{min poly of } J \]
\[ = \text{char poly of } J \]
\[ = \text{unique elem divisor of } J \]

The elem divisors of $T$ are

\[ f_1, f_2, \ldots, f_a \]

where $f_i = \text{elem divisor of } J_i$, $i = 1, \ldots, a$
Theorem 6

Referring to Theorem 4, suppose the VS $V$ has a second basis with which the matrix rep $T$ is

$$
T = \begin{pmatrix}
J_1' & & & \\
& J_2' & & \\
& & \ddots & \\
& & & J_s'
\end{pmatrix}
$$

where each $J_1', J_2', \ldots, J_s'$ is a Jordan block.

Then $S = e$ and $J_1', J_2', \ldots, J_s'$ is a perm $T$. $J_1', J_2', \ldots, J_s'$ is a perm $T$.

Proof

By Note 5 and the uniqueness of the s elem divisors for $T$. \qed
Ex 7

Given distinct $a, b, c \in F$

Assume $T$ has Jordan canonical form

Then the elem. divisors of $T$ are:

$x - a, \ (x - a)^2, \ (x - a)^3, \ \ (x - b)^2, \ (x - b)^2, \ (x - c)^2, \ (x - c)^3$
The irreducible factors of $T$ are

$$x - a,$$

$$\left(x - a\right)^2 \left(x - b\right)^2 \left(x - c\right)^2,$$

$$\left(x - a\right)^3 \left(x - b\right)^2 \left(x - c\right)^3.$$

The minimal poly of $T$ is

$$\left(x - a\right)^3 \left(x - b\right)^2 \left(x - c\right)^3.$$

The char poly of $T$ is

$$\left(x - a\right)^6 \left(x - b\right)^4 \left(x - c\right)^5.$$
DEF 8: Call $T$ **diagonalizable** whenever $V$ has a basis consisting of eigenvectors of $T$.

**Theorem 9:** With the above notation, **TFAE**

(i) $T$ is diagonalizable

(ii) The Jordan canonical form of $T$ is diagonal

(iii) Each Jordan block of $T$ is $1 \times 1$

(iv) The min poly of $T$ has no repeated roots

pf: reverse