Lecture 10

Motivation

Let $\mathcal{M} = \text{any vector space over } F$

Let $A = \text{any } F\text{-alg}$

An $A$-module $M$ on $\mathcal{M}$ is equivalent to an $F$-alg hom $A \to \text{End}(M)$

Given $A$-modules $M$, $N$

Consider tensor product

$$M \otimes N = M \otimes_F N$$

(so if $m_i$ is a basis for $M$ and $n_j$ is a basis for $N$ then

$$m_i \otimes n_j \text{ is basis for } M \otimes N$$

Wish to define an $A$-module structure on $M \otimes N$

Proceed as follows.

Observe that

$$A \otimes A$$

is an $F$-algebra with multiplication

$$(a \otimes a_1)(a_2 \otimes a_2') = (a_1 a_2) \otimes (a_1' a_2')$$

Observe that

$$M \otimes N$$

is an $A \otimes A$-module
with action

\[(a \circ a'), (m \circ a) = (a \circ m) \otimes (a \circ a')\]

This gives an $F$-alg hom

\[A \otimes A \to \text{End}(M \otimes N)\]

Suppose we can find an $F$-alg hom

\[A \to A \otimes A\]

Composing \(*\) \times \(*\) we get an $F$-alg hom

\[A \to \text{End}(M \otimes N)\]

Using this we get the desired $A$-module structure on $M \otimes N$

via our prelin comment, \[*\]
LEM 1: For unique $F$-alg hom $\Delta$:

$$\Delta: U_q \rightarrow U_q \otimes U_q.$$

From this, we have:

$$\Delta(e) = e \otimes 1 + k \otimes e,$$

$$\Delta(f) = f \otimes k + 1 \otimes f,$$

$$\Delta(k) = k \otimes k,$$

$$\Delta(k^\gamma) = k^\gamma \otimes k^\gamma.$$

We call $\Delta$ the "comultiplication" for $U_q$.

\textbf{pf} 

\textbf{Require:}

(i) \quad \Delta(k) \Delta(k^\gamma) = \Delta(k^\gamma) \Delta(k) = 1

(ii) \quad \Delta(k) \Delta(e) = \eta^2 \Delta(e) \Delta(k)

(iii) \quad \Delta(k) \Delta(f) = \eta^{-2} \Delta(f) \Delta(k)

(iv) \quad \Delta(e) \Delta(f) - \Delta(f) \Delta(e) = \frac{\Delta(k) - \Delta(k^\gamma)}{q - q^{-1}}$

Check (i).
\[ \text{check (1):} \]

\[ \text{LHS} = (k \bowtie k)(e \bowtie 1 + k \bowtie e) \]

\[ = k e \bowtie k + k^2 \bowtie e e \]

\[ = q e \bowtie k e + k \bowtie q e k \]

\[ = q^e (e \bowtie 1 + k \bowtie e)(k \bowtie k) \]

\[ = \text{RHS} \]

\[ \text{check (2):} \]

\[ \text{LHS} = k \bowtie 1 \]

\[ = k \}

\[ \square \]
Corollary 2. Let \( M, N \) denote \( U_q \)-modules. Then \( M \otimes N \) is a \( U_q \)-module with the following action:

\[
\forall x \in U_q \quad \forall m \in M \quad \forall n \in N
\]

\[
x \cdot (m \otimes n) = \sum_i \left( x_i \cdot m \right) \otimes (x_i^* \cdot n)
\]

where

\[
\Delta(x_i) = \sum_i x_i \otimes x_i^*.
\]

In particular,

\[
e \cdot (m \otimes n) = (e \cdot m) \otimes n + (k \cdot m) \otimes (e \cdot n)
\]

\[
f \cdot (m \otimes n) = (f \cdot m) \otimes (k^* \cdot n) + m \otimes (f \cdot n)
\]

\[
k \cdot (m \otimes n) = (k \cdot m) \otimes (k \cdot n)
\]

\[
k^* \cdot (m \otimes n) = (k^* \cdot m) \otimes (k^* \cdot n)
\]

pf By the previous discussion. \(\square\)
Given vector spaces $M_1, M_2, M_3$ and $F$

Recall

$$\left( M_1 \otimes M_2 \right) \otimes M_3$$

is

$$M_1 \otimes \left( M_2 \otimes M_3 \right)$$

An isomorphism is

$$(m_1 \otimes m_2) \otimes m_3 \rightarrow m_1 \otimes (m_2 \otimes m_3)$$

Call this the canonical isomorphism.

Next assume $M_1, M_2, M_3$ are $U_q$-modules.

By our previous comments each of $M_1, M_2, M_3$ has a $U_q$-module str. We want the canonical map to be an isomorphism of $U_q$-modules str. from $X$ to $X^X$.

This will occur if the map $\Delta: U_q \rightarrow M_1 \otimes M_2 \otimes M_3$ has the following property.
DEF 3 For an alg $A$, an algebra hom

\[ \Delta : A \to A \otimes A \]

is co-associative whenever the following diagram commutes:

\[\begin{array}{ccc}
A & \xrightarrow{\Delta} & A \otimes A \\
\downarrow{\Delta} & & \downarrow{\text{can}} \\
A \otimes A & \xrightarrow{\text{Id} \otimes \Delta} & A \otimes (A \otimes A) \\
\end{array}\]
LEMMA $\Delta : U_l \to U_l \otimes U_l$ is coassoc.

pf. We show the long commutes:

$$U_l \otimes U_l \to U_l \otimes (U_l \otimes U_l)$$

$$\Downarrow \Delta$$

$$U_l \otimes U_l \to (U_l \otimes U_l) \otimes U_l$$

$$\Delta \otimes I$$

For $x \in U_l$, we apply the various maps and check that results agree. Since each map is an alg hom., so it checks for $x \in e, t_k, k \in S$

$x = e$

$$e_0 + k_0 e \to e_0 (e_0) + k_0 (e_0 + k_0 e)$$

$$\Downarrow$$

$$e_0 (e_0) + k_0 (e_0 + k_0 e)$$

$$\Downarrow$$

$$(e_0) (e_0) + (k_0) (k_0) e$$

$$\Downarrow$$

$$(e_0 + k_0 e) \otimes I + (k_0 e) \otimes e$$

case of $f_k, u^2$ sim.
Lemma 5 \( \Delta : U^q \to U^q \otimes U^q \) satisfies

(i) \[ \Delta(e^n) = \sum_{r=0}^{n} q^{r(n-r)} \left[ \begin{array}{c} n \\ r \end{array} \right]_q e^{n-r} k^r \otimes e^r \]

(ii) \[ \Delta(f^n) = \sum_{r=0}^{n} q^{r(n-r)} \left[ \begin{array}{c} n \\ r \end{array} \right]_q f^r \otimes f^{n-r} k^{-r} \]

\( \forall n \in \mathbb{N} \)

Proof. By induction on \( n \). \( \square \)
LEM 6. \text{For} \; \text{any} \; \text{hom} \; \mathcal{E} : U_q \to IF \; \text{that sends}

\begin{align*}
    e \to 0, & \quad f \to 0, & \quad k \to 1, & \quad k^{-1} \to 1
\end{align*}

we call \( \mathcal{E} \) the \underline{\text{unit}} of \( U_q \).

pf routine \quad \Box

DEFF. Using \( \mathcal{E} \) above we view \( IF \) as a \( L \)-module \( U_q \)-module as follows:

\( \forall x \in U_q \) \quad \( \forall v \in IF \)

\[ x \cdot v = \mathcal{E}(x) v \]

We observe that \( e, f, k, k^{-1} \) vanish on this module.

We call \( IF \) the \underline{\text{trivial}} \; U_q \text{-module}.
Let $M$ denote any vector space.

Then

$$m \mapsto m \otimes 1$$

$$m \mapsto 1 \otimes m$$

is an iso of vs. Also

$$M \rightarrow IF \otimes M$$

Call $\otimes$ the Canonical isomorphisms.
Let $M$ denote a $U_1$-module.

View $F$ as a $U_1$-module as in ref. 7

Using $\Delta : U_1 \to U_1 \oplus U_1$ we get a $U_1$-module structure on $F \otimes M$ and $M \otimes F$.

We desire

Can: $M \to M \otimes F$ is iso of $U_1$-modules

Can: $M \to F \otimes M$ is iso of $U_1$-modules.

In order to motivate things we first consider a more general situation.
The trivial module — Motivation

Given algebra $A$

Given alg hom $\Delta : A \to A \otimes A$

Given alg hom $\varepsilon : A \to I$

Given $A$-module $M$

View $I$ as $A$-module via $\varepsilon$

View $I \otimes M$ as $A$-module via $\Delta$

We desire:

Can: $M \to I \otimes M$ is iso of $A$-modules.

Pick $x \in A$

Write $\Delta x = \sum_i x_i \otimes x'_i$, $x_i, x'_i \in A$

$$
\begin{array}{ccc}
M & \xrightarrow{\text{can}} & I \otimes M \\
m & \mapsto & 1 \otimes m \\
\downarrow & & \downarrow \\
x, m & \xrightarrow{\varepsilon(1 \otimes m)} & 1 \otimes (x, m)
\end{array}
$$

apply $x$
\[ 1 \otimes (x, \mu) = x, (1 \otimes \mu) \]
\[ = \Delta(x) (1 \otimes \mu) \]
\[ = \sum_i (x_i, 1) \otimes (x_i', \mu) \]
\[ = \sum_i \varepsilon(x_i) 1 \otimes (x_i', \mu) \]
\[ = 1 \otimes \left( \sum_i \varepsilon(x_i) x_i' \right), \mu \]

The desired conditions

\[ x = \sum_i \varepsilon(x_i) x_i' \]

This condition is expressed by saying that the following diagram commutes:

\[
\begin{array}{ccc}
\Delta & \to & A \otimes A \\
A & \downarrow & \downarrow \varepsilon \otimes 1 \\
I & & I \\
A & \to & F \otimes A
\end{array}
\]
Similarly, can \( M \to M \otimes F \) be iso of \( A \)-modules provided the following diagram commutes:

\[
\begin{array}{ccc}
A & \to & A \otimes A \\
\downarrow I & & \downarrow I \otimes E \\
A & \to & A \otimes F \\
\text{can}
\end{array}
\]