Lecture 20

The Hopf algebra $U_q(g)$

where $g = \text{f.d. s.s. Lie algebra}$

To motivate $U_q(g)$ for general $g$, first consider $g = \mathfrak{sl}_{n+1}$.

Recall Lie algebra $\mathfrak{sl}_{n+1}$

For moment assume $F = \mathbb{C}$

View $\mathfrak{sl}_{n+1} = \left\{ x \in \text{Mat}_{n+1}(F) \mid x^T x = 0 \right\}$

Lie bracket $[x, y] = xy - yx$ for $x, y \in \mathfrak{sl}_{n+1}$

Structure of $\mathfrak{sl}_{n+1}$ is described by Cartan matrix

\[ A = \begin{pmatrix} 2 & -1 & & \cdots & & \cdots & & \cdots & \cdots & & \cdots & \cdots \ 1 & 2 & -1 & & & & & & & & & & & & & & \end{pmatrix}_{n \times n} \]

As we now explain.
Notation

\[ F_n = \mathbb{R}_i, \text{ for } \}

\[ E_{ij} \in \text{Mat}_{mn}(\mathbb{F}) \]

has \((i, j)\) entry 1 and all other entries 0

Generators for linear span:

\[ e_i = E_{i\cdot\cdot n} \]

\[ f_i = E_{i,\cdot}\]

\[ h_i = E_{\cdot i} - E_{\cdot \cdot i\cdot} \]
(i) \( \left[ e_i, f_j \right] = \delta_{ij} h_i \quad \text{for all } i, j \)

(ii) \( \left[ h_i, h_j \right] = 0 \)

(iii) \( \left[ h_i, e_j \right] = A_{ij} e_j \)

(iv) \( \left[ h_i, f_j \right] = -A_{ij} f_j \)

(v) \( \left( \text{ad } e_i \right)^{1-A_{ij}} (e_j) = 0 \quad \text{if } i \neq j \)

(vi) \( \left( \text{ad } e_i \right)^{1-A_{ij}} (f_j) = 0 \quad \text{if } i \neq j \)

where \( \text{ad} x (y) = [x, y] \)

By Serre's Theorem, \( \mathfrak{sl}_n \) is isomorphic to the Lie algebra \( \mathfrak{gl}_n \) over \( \mathbb{F} \) generated by symbols \( e_i, f_i, h_i \) (\( i \in \mathbb{Z} \))

subject to relations (i) - (vi) above.

Also, the universal enveloping algebra \( \text{U}(\mathfrak{sl}_n) \) is the associative algebra with gens \( e_i, f_i, h_i \) (\( i \in \mathbb{Z} \)) and relations (i) - (vi) above, where as integer \( [xy] = xy - yx \).
Observations

1. For $\mathfrak{g}$ is even, $e_i, f_i, h_i$ span a Lie subalgebra of dimension 1000 of $\mathfrak{g}$.

2. Refer to (v) above, in $\mathfrak{u}(\mathfrak{sl}_n)$ the LHS is

$$
\sum_{a=0}^{1-A^{ij}} \left( \begin{array}{c}
1 - A^{ij} \\
-1
\end{array} \right) e_i^{a} e_j^{1-A^{ij}-a} e_i^{a}
$$
DEF \( \mathbb{F} \) an \( \mathbb{F} \)

\[ 0 \neq q \in \mathbb{F} \quad \text{not a root of 1} \]

For \( n \geq 1 \), the algebra \( U_q(\mathfrak{sl}_n) \) has\( k_i, k_i^{-1} \in \mathbb{F} \)

\[ e_i, f_i, k_i, k_i^{-1} \quad \text{is a gen} \]

and relations

(i) \( k_i k_i^{-1} = k_i^{-1} k_i = 1 \)

(ii) \( k_i k_j = k_j k_i \quad 1 \leq i < j \leq n \)

(iii) \( k_i e_j k_i^{-1} = q^{\delta_{ij}} e_j \)

(iv) \( k_i f_j k_i^{-1} = q^{-\delta_{ij}} f_j \)

(v) \( e_i f_j - f_j e_i = \delta_{ij} \left( \frac{k_i - k_i^{-1}}{q - q^{-1}} \right) \)

(vi) \( \sum_{a=0}^{1-Aq} \left[ \begin{array}{c} 1-Aq \\ a \end{array} \right] q^{\binom{a}{2}} e_i^{1-Aq^{-a}} f_j e_i^a = 0 \) if \( i \neq j \)

(vii) \( \sum_{a=0}^{1-Aq} \left[ \begin{array}{c} 1-Aq \\ a \end{array} \right] (q)^{a} f_i^{1-Aq^{-a}} f_j f_i^a = 0 \) if \( i \neq j \)
Next goal: Before \( U_1(g) \) for a fixed \( g \) is locally.

Needed facts on \( g \):

For now being, assume \( F = \mathbb{C} \).

Recall a f.d. Lie alg \( g \) over \( \mathbb{C} \) is semi-simple iff it is a direct sum of simple Lie algebras over \( \mathbb{C} \).

The simple Lie algebras over \( \mathbb{C} \) are:

\( A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 \)

Let \( g = \text{f.d. Lie alg} / \mathfrak{g} \).

We associate with \( g \) a Cartan matrix \( A \).

For \( g \) simple, \( A \) is given in handout.

For \( g = g_1 \oplus g_2 \oplus \cdots \oplus g_r \) (\( g_i \) simple),

\[
A = \begin{pmatrix}
A_1 & & & \\
& A_2 & & \\
& & A_3 & \\
& & & A_r
\end{pmatrix}
\]

\( A_i \) = Cartan matrix for \( g_i \).

By the rank of \( g \) we mean \( n \) where \( A \) is non-zero.
Proof of Theorem: The idea of the proof is to classify all the possible cobweb graphs. By a cobweb graph, we mean a graph whose vertices are pairs of integers, and whose edges connect vertices if the difference between the integers is a fixed positive integer. The cobweb graph is the dual graph to the graph of the roots of a polynomial. The roots of a polynomial can be classified into three types: real roots, complex roots, and roots of multiplicity greater than one.

Table 1: Cobweb matrices

The restrictions on $\Delta$ for types A, B, and C are imposed in order to avoid

Diagram 1: Cobweb graph

Diagram 2: Cobweb graph
Fact 1 \[ \det A > 0. \] In particular, \( A^{-1} \) exists.

Fact 2 \[ A \text{ is symmetrizable}, \quad \text{this means} \]
\[ \exists \text{pos. integers } \{ d_1, \ldots, d_n \}, \quad (n = \text{rank } A) \]
\[ \begin{pmatrix} d_1 & 0 \\ 0 & \ddots \\ & & d_n \end{pmatrix} A \]
\[ \text{is symmetric} \]

The \( d_i \) are not unique in gen.

We normalize the \( d_i \) so that

For \( q \) simple \[ 1 = \min \{ d_i \mid i \in q \} \]

For gen \( q \)

For each simple component of \( q \)
The entries \( d_i \) are normalized as above.
<table>
<thead>
<tr>
<th>Type</th>
<th>$d_1, d_2, \ldots, d_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n, B_n, C_n, E_6, E_7, E_8$</td>
<td>$l, 1, \ldots, 1$</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$1, 1, \ldots, 1, 2$</td>
</tr>
<tr>
<td>$C_n$</td>
<td>$2, 2, \ldots, 2, 1$</td>
</tr>
<tr>
<td>$F_4$</td>
<td>$1, 1, 2, 2$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$3, 1$</td>
</tr>
</tbody>
</table>
Fact 9

Let \( g = \mathfrak{h} \oplus \mathfrak{sl}_n \mathfrak{g} \) be a Lie algebra with \( \mathfrak{h} = \text{Cartan matrix} \) and \( n = \text{rank } \mathfrak{g} \).

Then \( \mathfrak{g} \) is isomorphic to the Lie algebra \( \mathfrak{g} \) with generators \( e_i, f_i, h_i \) with

\[ \text{and} \quad \gamma_2 \]

(\( i \)) \( [e_i, f_j] = \delta_{ij} h_i \) \( \quad \text{for } i, j \in \mathbb{N} \)

(\( ii \)) \( [h_i, h_j] = 0 \)

(\( iii \)) \( [h_i, e_j] = A_{ij} e_j \)

(\( iv \)) \( [h_i, f_j] = -A_{ij} f_j \)

(\( v \)) \( (\text{ad } e_i)^{1 - A_{ij}} (f_j) = 0 \) if \( i \neq j \)

(\( vi \)) \( (\text{ad } f_i)^{1 - A_{ij}} (e_j) = 0 \) if \( i \neq j \)

(\( vii \)) \( \text{ad} \times (\text{ad }) = [\eta, \eta] \)
Fact 5. Given a R.H. s.s. locally $\mathbb{C}$

\[
\exists \text{ nondeg symmetric bilinear form } g \times g \to \mathbb{C}
\]

such that

\[
([x, y], z) = (x, [y, z]) \quad \forall x, y, z \in g
\]

is not unique in $g_0$. It can be normalized so that

\[
(e_i, f_i) = \frac{1}{\delta_i} \quad \text{for } i \in \mathbb{N}_0
\]
For $g$ as above define

$$H = \text{Span}( h_1, h_2, \ldots, h_n )$$

Observe $H$ is a Lie subalgebra of $g$.

Call $H$ a **Cartan subalgebra**.

Let $H^* = \text{dual space of } H$.

We have a non-degenerate bilinear form

$$\langle , \rangle : H \times H^* \to \mathbb{C}$$

$$h \mapsto f(h)$$

**LEMMA 7.** For $g$, $H$, $H^*$ as above

$\exists$ unique basis $a_1, a_2, \ldots, a_n$

$\forall h_i \in H^*$

$$\langle h_i, a_j \rangle = A_{ij} \quad 1 \leq i, j \leq n$$

$A = \text{Cartan matrix}$

Call $a_1, a_2, \ldots$ the **simple roots** of $g$.

Write

$$\Pi = \{ a_1, a_2, \ldots, a_n \}$$

Proof: Since $A$ exists.
LEM 8. We have

(i) \[(h_1', h_0) = \frac{A_{ij}}{\partial x_j} = \frac{A_{ij}'}{\partial x_i} \quad \text{if } j \neq i\]

(ii) \[(h_i', h_i) = \frac{2}{\partial x_i} \quad \text{if } i \neq j \neq i\]

(iii) \[A_{ij} = \frac{2 (h_i', h_j)}{(h_j', h_j)} \quad \text{if } i \neq j \neq i\]

Proof (i) \[(h_1', h_2) = \left( [e_i, f_i], h_2 \right) = \left( e_i, [f_i, h_2] \right) = A_{ij} (e_i, f_i) = \frac{A_{ij}'}{\partial x_i} = \frac{A_{ij}'}{\partial x_j} \quad \text{by def of } A_{ij}\]

(ii) Set \( i = j \), \( A_{ii} = 2 \) in (i)

(iii) Conclude (i), (ii) \( \square \)
Corollary 9: The restriction of $\langle \cdot, \cdot \rangle$ to $H$ is non-degenerate.

Proof: By Lemma 8 (i) and since $A^*$ exists. \qed

The following map will be useful.

Lemma 10: For vector spaces $V \colon H \to H^y$

$\forall h \in H$

$\langle h', v(h) \rangle = (h', h)$ \hspace{1cm} $\forall h' \in H$

Proof: Each $\langle \cdot, \cdot \rangle$, $(\cdot)$ is non-deq. \qed
**Lemma 11** We have

\[ \nabla (h_i) = \frac{\partial_i}{d_i} \]

**Proof** In this step we check:

\[ \langle h_j, \frac{\partial_i}{d_i} \rangle = \langle h_j, \nabla (h_i) \rangle \]

\[ \| A_{2i} \| (h_j, h_i) \]

\[ \frac{A_{2i}}{d_i} \]

\[ \text{Lem 8 (i)} \]
Via \( \nu : H \to H^* \) we transport the bilinear form \((\cdot, \cdot)\) on \(H\), to a bilinear form \((\cdot, \cdot)\) on \(H^*\).

**Def 12.** There exists a bilinear form

\[
(\cdot, \cdot) : H^* \times H^* \to \mathbb{C}
\]

such that

\[
(x, \eta) = (\nu^{-1}(x), \nu^{-1}\eta) \quad \text{for all } x, \eta \in H^*
\]

We observe \((\cdot, \cdot)\) is symmetric and degenerate.