Lecture 27

Ref 59

Let

(i) \( U^+_1 = \text{subs} \cap U_1 \text{ gen by} \{ e^x \mid x \in \pi \} \)

(ii) \( U^-_1 = \ldots \)

(iii) \( U^0_1 = \ldots \)
Theorem 6.0 The following algebras are isomorphic:

(i) \( U_q^+ \)

(ii) The algebra generated by symbols \( \{ e_x / x \in \mathbb{R} \} \)

subject to the q-Serre relations:

\[ e_\alpha e_\beta + e_\beta e_\alpha = \frac{q^{|\alpha-\beta|} + q^{-|\alpha-\beta|}}{q^{|\alpha-\beta|} - 1} \quad e_\alpha^2 = 0 \]

\( \alpha, \beta \in \mathbb{R}, \alpha \neq \beta \)

pf Recall that the algebra \( \tilde{U}_q^+ \) is freely generated by \( \{ e_x / x \in \mathbb{R} \} \)

Recall \( I^+ \) is a two-sided ideal of \( \tilde{U}_q^+ \) generated by \( \{ e_{x+\alpha} / x, \alpha \in \mathbb{R} \} \)

So the algebra (ii) is iso

\[ \tilde{U}_q / I^+ \]

Recall our map \( \tilde{U}_q \rightarrow U_q \)

Let \( L = \text{ker} \)

Since the restriction

\[ \tilde{U}_q^+ \rightarrow U_q \]

is surjective, we find
\[ U_1^+ \text{ iso } U_9^+ \]

**Solution:**

\[ I^+ = L \cap \tilde{U}_q^+ \]

**Show:** \[ I^+ \leq L \cap \tilde{U}_q^+ \]

\[ I^+ \leq L \quad \text{since } q \text{-sine sub are away from } U_q^+ \]

\[ I^+ \leq \tilde{U}_1^+ \quad \text{by def} \]

**Show:** \[ I^+ \geq L \cap \tilde{U}_q^+ \]

Recall the mat map

\[ \tilde{U}_1^- \otimes \tilde{U}_9^- \otimes \tilde{U}_q^+ \rightarrow \tilde{U}_9^- \]

\[ a \otimes b \otimes c \rightarrow abc \]

\( C. \text{ iso of } v_5. \)
By Lemma 5.8, 

\[ L = \text{image under } (\ast) \text{ of } \]

\[ U_\gamma \ominus \tilde{U}_\gamma \otimes \mathbb{I}^+ + \mathbb{I}^- \otimes \tilde{U}_\gamma \otimes \tilde{U}_\gamma^+ \]

(\ast\ast)

Given \( x \in L \cap \tilde{U}_\gamma^+ \), show \( x \in \mathbb{I}^+ \)

let \( \tilde{x} = \text{preimage of } x \text{ under } (\ast) \)

Since \( x \in \tilde{U}_\gamma^+ \),

\[ \tilde{x} = 1 \otimes 1 \otimes x \]

\[ \in 1 \otimes 1 \otimes \tilde{U}_\gamma^+ \]

Since \( x \in L \),

\[ \tilde{x} \in (\ast\ast) \]

But since \( 1 \notin \mathbb{I}^- \) by construction,

\[ (\ast\ast) \cap 1 \otimes 1 \otimes \tilde{U}_\gamma^+ = 1 \otimes 1 \otimes \tilde{U}_\gamma^+ \]

Now \( 1 \otimes 1 \otimes x = \tilde{x} \in 1 \otimes 1 \otimes \tilde{U}_\gamma^+ \)

So \( x \in \mathbb{I}^+ \)

\[ \square \]
The following algebras are iso:

(i) \( \hat{U}_\eta \)

(ii) The algebra gen by symbols \( \{ f_\eta \mid x \in \pi \} \)

subject to the \( \eta \)-brane relations

\[ U_{\eta \beta} = 0, \quad \eta, \beta \in \pi, \quad \eta \neq \beta \]

\[ pf \sim to \#60. \]
Theorem 62

(i) The restriction of $\tilde{U}_\eta \to U_\eta$ to $\tilde{U}_\eta^0$ is a bijection $\tilde{U}_\eta^0 \to U_\eta^0$.

(ii) The vs $U_\eta^0$ has basis $\lambda \in \rho$.

(iii) The alg $U_\eta^0$ is iso of

$$
\tilde{\rho} \left[ \lambda_1^{\alpha_1} \cdots \lambda_k^{\alpha_k} \right]
$$

$\lambda_i, \alpha_i$ are cong indeps.

Proof:

Let $L = \ker f \text{ map } \tilde{U}_\eta \to U_\eta$

Suff to show

$L \cap \tilde{U}_\eta^0 = 0$

Recall mult map

$$
\tilde{U}_\eta \otimes \tilde{U}_\eta \otimes \tilde{U}_\eta \to \tilde{U}_\eta
$$

is iso of vs.
\[ L = \text{image under } \phi \] 

\[ \tilde{\mathbf{u}}_{\mathbf{g}} \otimes \tilde{\mathbf{u}}_{\mathbf{g}} \otimes I^+ + \mathbf{I} \otimes \tilde{\mathbf{u}}_{\mathbf{g}} \otimes \tilde{\mathbf{u}}_{\mathbf{g}}^+ \]  

\[(***)\]

Given \( x \in L \cap \tilde{\mathbf{u}}_{\mathbf{g}} \), show \( x = \alpha \)

Let \( \tilde{x} = \text{preimage of } x \text{ under } \phi \)

Since \( x \in \tilde{\mathbf{u}}_{\mathbf{g}} \),

\[ x = \lambda \alpha \phi \]

\[ \in \mathbf{I} \otimes \tilde{\mathbf{u}}_{\mathbf{g}} \otimes \mathbf{I} \]

Since \( x \in L \),

\[ \tilde{x} \in \mathbf{X} \mathbf{X} \]

Since \( 1 \in I^+ \) \( , \) \( 1 \notin I^- \),

\[ (**) \land \mathbf{I} \otimes \tilde{\mathbf{u}}_{\mathbf{g}} \otimes \mathbf{I} = 0 \]

Now \( \lambda \alpha \phi \phi = \tilde{x} = 0 \)

So \( x = 0 \)

\[(iii), (iii) \) \( B_1 \) \( (ii) \) and \( \text{Lem 98} \)

\( \square \)
Theorem 6.3

The map

\[ U_q^- \otimes U_q^o \otimes U_q^+ \rightarrow U_q \]

\[ a \otimes b \otimes c \rightarrow abc \]

is an isomorphism of vector spaces.

Proof

The quotient map

\[ \sigma : \tilde{U}_q \rightarrow U_q \]

is an algebra homomorphism.

We have

\[ \sigma^+ = \text{restriction of } \sigma |_{\tilde{U}_q^+} \]

\[ \sigma^o = \text{restriction of } \sigma |_{\tilde{U}_q^o} \]

\[ \sigma^- = \text{restriction of } \sigma |_{\tilde{U}_q^-} \]

we see

\[ I^+ = \text{kernel of } \sigma^+ : \tilde{U}_q^+ \rightarrow U_q^+ \]

\[ \sigma^o : \tilde{U}_q^o \rightarrow U_q^o \]

\[ \sigma^- : \tilde{U}_q^- \rightarrow U_q^- \]
Consider the diagram:

\[ \tilde{U}_\gamma \otimes \tilde{U}_\gamma \otimes \tilde{U}_\gamma \]

\[ \sigma \otimes \sigma \otimes \sigma^+ \]

Show that the diagram commutes:

\[ a \otimes b \otimes c \]

\[ \sigma(a) \otimes \sigma(b) \otimes \sigma(c) \]

Show \( \Phi \) is injective.

Given \( x \in K \) show \( x = 0 \).
let \( \tilde{x} = \text{preimage of } x = \tilde{u}_- \otimes \tilde{u}_0 \otimes \tilde{u}_+ \)

under \( \sigma^- \otimes \sigma^0 \otimes \sigma^+ \) is

\[ \tilde{x} \]

\[ \downarrow \]

\[ x \rightarrow 0 \]

Image of \( \tilde{x} \) under \( \text{mult} \) is in \( \ker \sigma^0 \)

So \( \tilde{x} \in \tilde{u}_- \otimes \tilde{u}_0 \otimes I^+ + I^- \otimes \tilde{u}_0 \otimes \tilde{u}_+ \)

by LEM 58.

But

\[ \sigma^- \otimes \sigma^0 \otimes \sigma^+ \left( \tilde{u}_- \otimes \tilde{u}_0 \otimes I^+ \right) \]

\[ = \sigma^{- \left( \tilde{u}_- \right)} \otimes \sigma(\tilde{u}_0) \otimes \sigma(\tilde{I}_+) \]

\[ = 0 \]
and similarly

\[ \sigma^- \otimes \sigma^0 \otimes \sigma^+ \left( \mathbb{1} \otimes \tilde{U}_x \otimes \tilde{U}_y \right) = 0 \]

So

\[ \sigma^- \otimes \sigma^0 \otimes \sigma^+ \left( \tilde{x} \right) = 0 \]

\[ \tilde{x} \]

We have shown \( \tilde{A} \) is iso.

\( \tilde{A} \) is surjective by construction, so \( \tilde{A} \) is an iso.
LEM 69

(i)  \( \exists \) unique alg hom

\[ \omega: \tilde{u}_1 \to \tilde{u}_1 \quad \text{(map } \omega: u_1 \to u_1) \]

that sends

\[ e_x \to f_x \]

\[ f_x \to e_x \]

\[ k_x \to k_x \tilde{f}_1 \]

Morcan \( \omega^x = 1 \).

(ii)  \( \exists \) unique anti-\text{aut}

\[ \tau: \tilde{u}_1 \to \tilde{u}_1 \quad \text{(map } \tau: u_1 \to u_1) \]

that sends

\[ e_x \to e_x \]

\[ f_x \to f_x \]

\[ k_x \to k_x \tilde{f}_1 \]

Morcan \( \tau^x = 1 \).

pf Routine
LEM 6.5 \quad E_n \propto E_T

(i) The elements \( \{ e^z \mid z \in \mathbb{R} \} \) are linearly independent in \( U_2 \).

(ii) \[ \{ e^{iz} \mid z \in \mathbb{R} \} \quad \cdots \]

pf. By \( n_{60}, n_{61}, n_{63} \) \qed
LEMMA 66

\[ F_\alpha \ni \delta \rightarrow \alpha \]

\[ U_{\gamma} ( \alpha \lambda \beta ) \rightarrow U_\gamma \]

\[ e \rightarrow e_\alpha \]

\[ f \rightarrow f_\alpha \]

\[ k^{\pm 1} \rightarrow k^{\pm 1} \]

is injective.

pf: By Lemma 63 and Lemma 65.