1 The Adjoint Case for $B_2$

Here we will describe the quantum adjoint representation for the cases where $\Phi$ is of rank 2. Specifically I will describe it when $\Phi$ is type $B_2$ and $G_2$ because the $A_2$ case has already been done and the $A_1 \times A_1$ case is rather trivial. First, $B_2$. We can write $\Pi = \{\alpha, \beta\}$ with $\alpha$ as the long root and $\beta$ as the short root. The Cartan matrix of $B_2$ is
\[
\begin{pmatrix}
2 & -2 \\
-1 & 2
\end{pmatrix}
\] so $\langle \alpha, \beta \rangle = -2$ and $\langle \beta, \alpha \rangle = -1$.

*Draw picture of $B_2$* It is not too hard to see that the largest root in this set is $\alpha + 2\beta$, so what we are going to describe here is $L = L(\alpha + 2\beta)$. Also, each weight of $L$ is conjugate under the action of the Weyl group to a dominant root less than or equal to $\alpha + 2\beta$, which means that our weights are all in $\Phi \cup \{0\}$. We can also say that the dimension of $L_\gamma$ for $\gamma \in W(\alpha + 2\beta) = \{\pm(\alpha +)\}$. Stuff about $L_\gamma$.

Let $x_{\alpha + 2\beta}$ be a basis vector for $L_{\alpha + 2\beta}$.

First let us write for all $\lambda \in \Lambda$
\[
L_\lambda = F_\alpha L_{\lambda + \alpha} + F_\beta L_{\lambda + \beta}
\]

Plugging in $\alpha + \beta$ here for $\lambda$ gives us that
\[
L_{\alpha + \beta} = F_\beta L_{\alpha + 2\beta} + F_\beta L_{2\alpha + \beta} = k(F_\beta x_{\alpha + 2\beta})
\]

because $x_{\alpha + 2\beta}$ is the basis for $L_{\alpha + 2\beta}$ and $L_{2\alpha + \beta} = 0$ ($2\alpha + \beta \notin \Phi$).

Here I will compute $E_\beta F_\beta x_{\alpha + 2\beta}$, partially to show the computation, and partially for reasons I will explain in a minute. We have that
\[
E_\beta F_\beta x_{\alpha + 2\beta} = F_\beta E_\beta x_{\alpha + 2\beta} + \frac{K_\beta - K_\beta^{-1}}{q_\beta - q_\beta^{-1}} x_{\alpha + 2\beta}
\]

\[
= 0 + \frac{K_\beta q_\beta - K_\beta^{-1} q_\beta^{-1} x_{\alpha + 2\beta}}{q_\beta - q_\beta^{-1}}
\]

\[
= 0 + \frac{K_\beta q_\beta - K_\beta^{-1} q_\beta^{-1}}{q_\beta - q_\beta^{-1}} x_{\alpha + 2\beta}
\]

\[
= \frac{q^2 - q^{-2}}{q - q^{-1}} x_{\alpha + 2\beta}
\]

\[
= \frac{q^2 - q^{-2}}{q - q^{-1}} x_{\alpha + 2\beta}
\]

because $\beta$ short and so $q_\beta = q^{d_\beta} = q^{(\beta, \beta)/2} = q$

\[
= \frac{q^2 - q^{-2}}{q - q^{-1}} x_{\alpha + 2\beta}
\]

\[
= \frac{q^2 - q^{-2}}{q - q^{-1}} x_{\alpha + 2\beta}
\]

There will be many other similar computations of this sort, so I will omit them. They tend to differ only by something in a bracket product, which might not even make a difference. This computation is important because it shows that because $[2]_\beta = q_\beta + q_\beta^{-1} \neq 0$ as $q$ is not a root of unity, that $F_\beta x_\alpha + 2\beta \neq 0$. Given our earlier formula for $L_{\alpha + \beta}$ this implies that the dimension of $L_{\alpha + \beta}$ and $L_\gamma$ for $\gamma \in W(\alpha + \beta)$ is one. Let us set
\[
x_{\alpha + \beta} = F_\beta x_{\alpha + 2\beta}
\]

and
\[
x_\alpha = \frac{1}{[2]_\beta} F_\beta x_{\alpha + \beta}.
\]
Applying $E_\beta$ to each of these gives

$$E_\beta x_{\alpha + \beta} = [2]_\beta x_{\alpha + 2\beta}$$

and

$$E_\beta x_\alpha = x_{\alpha + \beta}$$

It is easy to see that the $x_{\alpha + \beta}$ and $x_\alpha$ are in $L_{\alpha + \beta}$ and $L_\alpha$. These equations also tell us that $x_{\alpha + \beta}$ and $x_\alpha$ are nonzero, and thus that the do actually form bases for $L_{\alpha + \beta}$ and $L_\alpha$. I will be defining many more basis elements in similar ways, and the same logic can be used to show that they are indeed basis elements. This gives us two more pieces in describing how the $F$’s and $E$’s act on this module. Now let us set

$$x_\beta = F_\alpha x_{\alpha + \beta}$$

Applying by $E_\beta$ and doing a similar computation to the one we did earlier gives us that

$$E_\alpha x_\beta = x_{\alpha + \beta}$$

2 \hspace{1cm} L_0

We will now move on to describing $L_0$, the only multidimensional weight space of $L$. From our equation earlier we have that

$$L_0 = F_\alpha L_\alpha + F_\beta L_\beta = k(F_\alpha x_\alpha) + k(F_\beta x_\beta)$$

so we can set $h_\alpha = F_\alpha x_\alpha$ and $h_\beta = F_\beta x_\beta$ so that $h_\alpha, h_\beta$ span $L_0$. Now set

$$x_{-\alpha} = \frac{1}{[2]_\alpha} F_\alpha h_\alpha$$

and

$$x_{-\beta} = \frac{1}{[2]_\beta} F_\beta h_\beta$$

We can apply $E_\alpha$ to the first of these equations and apply $E_\beta$ to the second

$$E_\alpha x_{-\alpha} = h_\alpha$$

and

$$E_\beta x_{-\beta} = h_\beta$$

These calculations are similar to the ones we performed earlier, and give us another part of our description for this module. Another set of similar calculation, this time done by applying $E_\alpha$ and $E_\beta$ to the definitions for $h_\alpha$ and $h_\beta$ respectively gives

$$E_\alpha h_\alpha = [2]_\alpha x_\alpha$$

and

$$E_\beta h_\beta = [2]_\beta x_\beta$$

which then become another part of our description of $L$. Now, computing $E_\alpha h_\beta$ and $E_\beta h_\alpha$ is easy because

$$E_\alpha h_\beta = E_\alpha F_\beta x_\beta = F_\beta E_\alpha x_\beta = F_\beta x_{\alpha + \beta} = [2]_\beta x_{\alpha}.$$
The same sort of calculation gives that $E_\beta h_\alpha = x_\beta$.

Now we do not automatically know that $h_\alpha$ and $h_\beta$ form a basis for $L_0$, only that they generate it. However, due to the formulas just derived, $E_\beta h_\beta = [2]_\beta E_\beta h_\beta$, so if $h_\beta$ and $h_\alpha$ are no linearly independent then $h_\beta = [2]_\beta h_\alpha$. Applying $E_\alpha$ to this equation results in $[2]_\beta x_\alpha = [2]_\beta [2]_\alpha x_\alpha$, which in turn implies that $1 = [2]_\alpha = q_\alpha + q_\alpha^{-1}$, which can only happen if $q$ is a root of unity. - Argument that $h_\alpha$ and $h_\beta$ are linearly independent.

Now we are going to try to compute $F_\alpha h_\beta$ and $F_\beta h_\alpha$. To do this we need to employ the quantum Serre relations, which state that

$$\sum_{s=0}^{1-a_{\alpha\beta}} (-1)^s \left[ \frac{1-a_{\alpha\beta}}{s} \right] F_\alpha^{1-a_{\alpha\beta}-s} F_\beta F_\alpha^s = 0.$$ 

Because $a_{\alpha\beta}$ equals either -1 or -2 we get two versions of this relation, one of which $(a_{\alpha\beta} = -1)$ we can apply to $x_{\alpha+\beta}$ to get

$$F_\alpha^2 F_\beta x_{\alpha+\beta} - [2]_{\alpha} F_\alpha F_\beta F_\alpha x_{\alpha+\beta} + F_\beta F_\alpha^2 x_{\alpha+\beta} = 0$$

Now we know that $F_\alpha x_{\alpha+\beta} = x_\beta$, that $F_\beta x_\beta = h_\beta$, that $F_\beta x_{\alpha+\beta} = [2]_\beta x_\alpha$, that $F_\alpha x_\alpha = h_\alpha$ and that $F_\alpha h_\alpha = [2]_\alpha x_{\alpha-\alpha}$, so we can simplify this way down into

$$[2]_\alpha [2]_\beta x_{\alpha-\alpha} - [2]_\alpha F_\alpha h_\beta = 0$$

which can be simplified down to

$$F_\alpha h_\beta = [2]_\beta x_{\alpha-\alpha}.$$ 

Applying the relation in the case where $a_{\alpha\beta} = -2$ to $x_{\alpha+2\beta}$ gives a similar equation with 4 terms. We then know that some of them get mapped out of $L_\gamma$ for $\gamma \in \Phi$, so they are zero, and we can reduce the others with formulas we already know to get

$$[3]_\beta [2]_\beta x_{\gamma-\gamma} + [3]_\beta [2]_\beta F_\beta h_\alpha = 0$$

which then gives us

$$F_\beta h_\alpha = x_{\gamma-\gamma}.$$ 

Now, let us set

$$x_{-(\alpha+\beta)} = F_\beta x_{\alpha-\alpha}$$

and

$$x_{-(\alpha+2\beta)} = \frac{1}{[2]_\beta} F_\gamma x_{\alpha+2\beta}$$

We can do a few things to these to finish our description of $L$. Applying $E_\beta$ to both of these gives

$$E_\beta x_{-(\alpha+2\beta)} = x_{-(\alpha+\beta)}$$

and

$$E_\beta x_{-(\alpha+\beta)} = [2]_\beta x_{\alpha-\alpha}.$$ 

Applying $E_\alpha$ to the first gives

$$E_\alpha x_{-(\alpha+\beta)} = E_\alpha F_\beta x_{\alpha-\alpha} = F_\beta E_\alpha x_{\alpha-\alpha} = F_\beta h_\alpha = x_{\gamma-\gamma}.$$ 

Finally, going back to the definition of $x_{\gamma-\gamma} = \frac{1}{[2]_\beta} F_\gamma h_\beta$, we can see that

$$F_\alpha x_{\gamma-\gamma} = x_{-(\alpha+\beta)}.$$ 

Any application of a generator to a basis element for $L_\gamma$ which 1 have not given a formula for is equal to zero, because the basis element gets sent outside $L_\gamma$ for $\gamma \in \Phi \cup \{0\}$. Also applying $K_\mu$ to any of our basis vectors is easy because they are weight vectors.
3 The Adjoint Case for $G_2$

We can do a very similar thing here as we did when $\Phi$ is of type $G_2$. For $\alpha$ long and $\beta$ short we instead have that $\langle \alpha, \beta \rangle = -3$ and $\langle \beta, \alpha \rangle = -\frac{3}{2}$, and the positive roots of $\Phi$ are $\alpha, \beta, \alpha + \beta, \alpha + 2\beta, \alpha + 3\beta, 2\alpha + 3\beta$. The largest of these is $2\alpha + 3\beta$, so we are going to describe $L(2\alpha + 3\beta)$. First, let $x_{2\alpha + 3\beta}$ be a basis vector for $L_{2\alpha + 3\beta}$. Then, let

$$x_{\alpha + 3\beta} = F_{\alpha} x_{2\alpha + 3\beta}.$$ 

By applying $E_{\alpha}$ we get

$$E_{\alpha} x_{\alpha + 3\beta} = x_{2\alpha + 3\beta}.$$ 

Now we define

$$x_{\alpha + 2\beta} = F_{\beta} x_{\alpha + 3\beta}$$

and

$$x_{\alpha + \beta} = \frac{1}{[2]_\beta} F_{\beta} x_{\alpha + 2\beta}$$

and

$$x_{\alpha} = \frac{1}{[3]_\beta} F_{\beta} x_{\alpha + \beta}.$$ 

Applying $E_{\beta}$ to each of these results in the formulas

$$E_{\beta} x_{\alpha + 2\beta} = [3]_\beta x_{\alpha + 3\beta}$$

and

$$E_{\beta} x_{\alpha + \beta} = [2]_\beta x_{\alpha + 2\beta}$$

and

$$E_{\beta} x_{\alpha} = x_{\alpha + \beta}.$$ 

Again the computations here are just the same as before, just with differences determined by the values of our roots taken in our bracket operator. Now if we let $x_{\beta} = F_{\alpha} x_{\alpha + \beta}$, then another similar computation gives that

$$E_{\alpha} x_{\beta} = x_{\alpha + \beta}.$$ 

3.1 $L_0$

Now we can work to describe the action of our generators on a basis for $L_0$. For the same reason as in the $B_2$ case, we can set

$$h_{\alpha} = F_{\alpha} x_{\alpha}$$

and

$$h_{\beta} = F_{\beta} x_{\beta}$$

to get a basis for $L_0$. Applying $E_{\alpha}$ to the first and $E_{\beta}$ to the second gives

$$E_{\alpha} h_{\alpha} = [2]_\alpha x_{\alpha}$$

and

$$E_{\beta} h_{\beta} = [2]_\beta x_{\beta}.$$ 

We can also set

$$x_{-\alpha} = \frac{1}{[2]_\alpha} F_{\alpha} h_{\alpha}$$
and

\[ x_{-\beta} = \frac{1}{[2]_{\beta}} F_{\beta} h_{\beta}. \]

Applying \( E_{\alpha} \) to the first and \( E_{\beta} \) to the second gives

\[ E_{\alpha} x_{-\alpha} = h_{\alpha}. \]

and

\[ E_{\beta} x_{-\beta} = h_{\beta}. \]

We can add all of these to what we know about how the generators of \( U_q \) act on \( L \). Similarly to how we did a few times for \( B_2 \), we can apply \( E_{\beta} \) and \( E_{\alpha} \) to the formulas for \( h_{\alpha} \) and \( h_{\beta} \) respectively to get

\[ E_{\alpha} h_{\beta} = [3]_{\beta} x_{\alpha} \]

and

\[ E_{\beta} h_{\alpha} = x_{\beta}. \]

A similar argument as in the \( B_2 \) case shows that \( h_{\alpha} \) and \( h_{\beta} \) are linearly independent and thus a basis for \( L_0 \). We will now diverge slightly from what we did in the \( B_2 \) case to determine \( F_{\alpha} h_{\beta} \) and \( F_{\beta} h_{\alpha} \) by not using the \( q \)-Serre relations. Multiplying the equations \( E_{\alpha} h_{\alpha} = [2]_{\alpha} x_{\alpha} \) and \( E_{\beta} h_{\beta} = [3]_{\beta} x_{\alpha} \) by \([3]_{\beta}\) and \([2]_{\alpha}\) respectively shows that

\[ [2]_{\alpha}[3]_{\beta} x_{\alpha} = E_{\alpha} h_{\alpha} [3]_{\beta} = E_{\alpha} h_{\beta} [2]_{\alpha} \]

and so

\[ E_{\alpha} \left( h_{\beta} - \frac{[3]_{\beta}}{[2]_{\alpha}} h_{\alpha} \right) = 0. \]

We can then apply \( F_{\alpha} \) to these to get

\[ E_{\alpha} F_{\alpha} \left( h_{\beta} - \frac{[3]_{\beta}}{[2]_{\alpha}} h_{\alpha} \right) + \frac{K_{\alpha} - K_{\alpha}^{-1}}{q_{\alpha} - q_{\alpha}^{-1}} \left( h_{\beta} - \frac{[3]_{\beta}}{[2]_{\alpha}} h_{\alpha} \right) = 0 \]

by (R4). The second term of this sum is just zero, because \( K_{\alpha} h_{\beta} = h_{\beta} \) and \( K_{\alpha} h_{\alpha} = h_{\alpha} \) because both \( h_{\alpha} \) and \( h_{\beta} \) are in \( L_0 \). The first term can only be zero if \( F_{\alpha}(h_{\beta} - (3\beta/[2]_{\alpha})h_{\alpha}) = 0 \) so we can say that

\[ F_{\alpha} \left( h_{\beta} - \frac{[3]_{\beta}}{[2]_{\alpha}} h_{\alpha} \right) = 0. \]

Expanding this gives

\[ F_{\alpha} h_{\beta} = \frac{[3]_{\beta}}{[2]_{\alpha}} F_{\alpha} h_{\alpha} = [3]_{\beta} x_{-\alpha}. \]

We can do a similar trick with the equations \( E_{\beta} h_{\alpha} = x_{\beta} \) and \( E_{\beta} h_{\beta} = [2]_{\beta} x_{\beta} \) which show that

\[ E_{\beta} (h_{\beta} - [2]_{\beta} h_{\alpha}) = 0. \]

Applying \( F_{\beta} \) and expanding gives

\[ F_{\beta} h_{\alpha} = x_{-\beta}. \]

Continuing on to other parts of \( L \), let

\[ x_{-(\alpha+\beta)} = F_{\beta} x_{-\alpha} \]
\[ x_{-(\alpha + 2\beta)} = \frac{1}{[2]_\beta} F_{\beta} x_{-\alpha + \beta} \]

and

\[ x_{-(\alpha + 3\beta)} = \frac{1}{[3]_\beta} F_{\beta} x_{-(\alpha + 2\beta)} \]

Applying \( E_\beta \) to each of these gives

\[ E_\beta x_{-(\alpha + \beta)} = [3]_\beta x_{-\alpha} \]

\[ E_\beta x_{-(\alpha + 2\beta)} = [2]_\beta x_{-(\alpha + \beta)} \]

and

\[ E_\beta x_{-(\alpha + 3\beta)} = x_{-(\alpha + 2\beta)} \cdot \]

Now, let us write

\[ E_\beta x_{-(\alpha + \beta)} = [3]_\beta x_{-\alpha} = F_\alpha h_\beta. \]

We can rewrite this as

\[ F_\alpha E_\beta x_{-\beta} = E_\beta x_{-(\alpha + \beta)} \]

which by (R4) can be again rewritten as

\[ E_\beta F_\alpha x_{-\beta} = E_\beta x_{-(\alpha + \beta)} \]

so

\[ F_\alpha x_{-\beta} = x_{-\alpha + \beta}. \]

We can finally set

\[ x_{-(2\alpha + 3\beta)} = F_\alpha x_{-(\alpha + 3\beta)} \]

which upon applying \( E_\alpha \) we get

\[ E_\alpha x_{-(2\alpha + 3\beta)} = x_{-(\alpha + 3\beta)} \cdot \]

Again, any formula not given for a generator applied to a basis element of \( L_\gamma \) for some \( \gamma \in \Phi \cup \{0\} \) is just 0, because the generator sends the basis element to some \( L_\gamma \) for some \( \gamma \notin \Phi \cup \{0\} \). The action of any \( K_\mu \) on some basis element is also well understood because each basis element is a weight vector we know the weight for.