Hello everyone. My intention for today was to follow a paper from 2005 by Phung Ho Thi entitled "In the henselian extensions of henselian groups of type A." Though the first section was difficult, it was the most straightforward. This, however, proved to be absorbing, as their paper depended largely on algebraic results largely different from anything covered this semester and understanding these statements still results in a level of clarity a snap of these concepts.

Furthermore, straightforward examples of these (significantly different) preliminary and algebraic results some cited by her and some not of considerable interest which helped me understand the paper under our consideration today, and would like to share these with you.

Her paper's motivations are the Henselian and, crucially related, the Hecke algebra. Today I will give an overview of these tools as she has revised it. Her paper, with discussion will I hope, be of interest to those among you interested in reading the paper in greater depth.
So to begin, let us define the Hecke symmetry as (formally) the half way (in some 
irreducibility) at the beginning of the first lecture of the introduction of this paper. 
Some assumptions throughout:

- We are working over an algebraically closed field \( k \) of characteristic 0.
- \( V \) is a vector space over \( k \) of dimension \( d \).

**Def.** Let \( R : V \otimes V \to V \otimes V \) be an invertible operator. \( R \) is a

Hecke symmetry if the following are fulfilled:

- \( R_1 R_2 R_1 = R_2 R_1 R_2 \) where \( R_1 := R \otimes \text{Id}_V \); \( R_2 := \text{Id}_V \otimes R \) (Yang-Baxter equation)
- \((R+I)(R-q) = 0\) for some \( q \in k \) (Hecke equation)
- The half adjoint to \( R \), \( R^\dagger \), is invertible.

(Note: the order for the first 2 equations are not in the proper order.

**Remark:** In a previous paper of 1991, which is referenced in [1].

**Remark:** There are various definitions of \( R^\dagger \), I find that the most accessible

is that which studies its matrix representation (although this does obscure

some of \( R^\dagger \)'s motivation).

Let us fix a basis \( \{ x_1, x_2, \ldots, x_d \} \) of \( V \). Then \( R \) can be

given by a matrix (else denoted \( R \)) as

\[
R(x; \otimes x_j) = \sum_{k,l} x_k \otimes x_l R_{kl}^{x_j}
\]
Now, let \( e_1, e_2, \ldots, e_d \) be a basis for the dual space \( V^* \), where \( e_i(x_j) = \delta_{ij} \), as is standard.

**Def.** \( R^* \) is a function from \( V^* \otimes V \to V \otimes V^* \) such that

\[
R^*(e_i \otimes x_j) = \sum_{k,l} x_k \otimes e_l R_{ik}.
\]

Therefore, the invertibility conditions of \( R^* \) can be expressed as follows:

There exists a matrix \( P \) such that

\[
\sum_{m} P_{im} R_{mk} = \delta_{is} \delta_{tk} \quad \text{and} \quad \sum_{m} R_{jm} P_{mk} = \delta_{js} \delta_{ik}.
\]

The proof is left as an exercise.

Before proceeding to How's second Hecke symmetry, we must investigate a couple of other concepts.

Let \( GL(V) \) be the group (inner product \( V \)) as usual. Recall from your knowledge of algebra that \( GL(V) \) acts on the \( i \)th homogeneous component of the exterior algebra over \( V \) by means of the determinant. More precisely, \( X_d(V) \) is 1-dimensional and a non-zero basis vectors \( x_1 \wedge x_2 \wedge \cdots \wedge x_d \). If \( g \in GL(V) \) has the matrix \( A \) (with respect to the given basis), then

\[
- g \cdot (x_1 \wedge x_2 \wedge \cdots \wedge x_d) := (g(x_1) \wedge g(x_2) \wedge \cdots \wedge g(x_d))
\]

\[
= \det(A)(x_1 \wedge x_2 \wedge \cdots \wedge x_d).
\]
Next, we extend the concept of a super space.

**Def.** A vector superspace $V$ is a $\mathbb{Z}_2$-graded vector space with decomposition

$$V = V_0 + V_1, \quad 0, 1 \in \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}.$$  

Vectors that are elements of either $V_0$ or $V_1$ are said to be homogeneous.

The parity of a non-zero homogeneous element $x$, $|x|$, is $0$ if $x \in V_0$ and $1$ if $x \in V_1$.

If $V_0$ and $V_1$ have dimensions $r$ and $s$ respectively, we say $V$ is a superspace of dimension $(r | s)$.

Now let $V$ be a vector superspace of dimension $(r | s)$, with $r + s = d$.

Let $\{x_1, x_2\}$ and be a homogeneous basis of $V$ (i.e., a basis made of homogeneous elements) in which the first $r$ elements have even parity and the remaining $s$ elements have odd parity.

Let $z^i$ be an endomorphism of $V$ such that $z^i(x_k) = x_{s+k}$.

The super-group of endomorphisms that commute with $z^i$ $GL(V)$ is generated by the set $z^i$. As $V$ is a $\mathbb{Z}_2$-graded vector space, $GL(V)$ has a $\mathbb{Z}_2$-grading:

$$|z^i| = |x_{s+k}| = 1|x_k| + 1|x_{s+k}| (s + 2) = \mathbb{Z}_2.$$  

Thus, an element of $GL(V)$ can be uniquely represented by a super matrix whose invertibility can be given in terms of a super determinant called the Berezinian.
If we have \( Z \in \mathcal{GL}(V) \) and
\[
Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]
where \( A, D \) are square matrices of dimensions \( mxm \) and \( nxn \), respectively, and \( \text{Whitney's parity} \) is even, and \( B, D \) are matrices of the dimensions \( mnx \) and \( nmx \) whose parities are odd, the superdeterminant is defined to be:
\[
\text{Ber}(Z) = (\det D)^{-1} \det(A - CD^+B).
\]

Prove that the product \( Z \) is invertible if and only if its superdeterminant \( \text{Ber}(Z) \) is invertible, and that the superdeterminant is multiplicative:
\[
\text{Ber}(Z_1Z_2) = \text{Ber}(Z_1) \text{Ber}(Z_2).
\]

From this it is clear that \( \mathcal{GL}(V) \) is a supergroup, i.e., a group with \( \mathbb{Z}_2 \)-grading.

Now, let us return to our discussion of Hecke symmetries. Consider the tensor algebra:
\[
E_k := \mathbb{Z}_2 < \{ E^{ij}_k \}_{i,j=1}^n > / \left( E^{ij}_k E^{kl}_m = R^{ij}_{mk} E^{kl}_m + E^{lk}_m E^{ij}_k \right).
\]
There is a natural action of \( E_k \) on \( V \), given by the extension of the definition of \( E^{ij}_k \). By the usual properties of linear maps,
\[
\text{this is a vector } \bigotimes V \otimes E_k \rightarrow V. \text{ The dual of this map is } E_k \end{eqnarray*}
\]
\[
\text{The dual of this map is } E_k \end{eqnarray*}
This construction is a conclusion of \( \mathcal{V} \).

Now, we say that the Hecke Algebra \( \mathcal{H}_N \) is defined as

\[
\mathcal{H}_N = \mathcal{H}_N \quad \text{for} \quad J
\]

\[
k \langle \{ T_i \} \rangle_{i \leq N} \bigg/ \left( T_i T_j = T_j T_i, \quad |i-j| \geq 2 \right) \quad T_i T_i = T_i T_i T_i
\]

\[
T_i^2 = (q-1) T_i + q.
\]

The Hecke symmetry is given by action of \( \mathcal{H}_N \) on \( \mathcal{V}^n \) by identifying \( \overset{\square}{T_i} \) with \( R_f = \text{id} + \Theta \sigma \Theta^{-1} \).

It is the whole point that the commutativity of \( \mathcal{H}_N \) is not mere formal. The following "Doublemation Theorem":

(From the algebras \( \mathcal{H}_N \) and \( \mathcal{E}_\mathcal{J} \) are centralizers of each other in)

\[
\mathcal{E}_{d+1}(\mathcal{V}^n).
\]
The primary elements at the base of which architecture is founded by practicing

Theorem: The sum of the sides in a triangle is greater than the height of the perpendicular from the vertex opposite the base.

\[ a + b > h \]

\[ c + d > e \]

\[ x + y > z \]

\[ a + b + c + d + e = f + g + h + i + j \]

\[ k + l = m + n \]

where \( \alpha, \beta, \gamma, \delta, \theta, \phi, \xi, \psi, \omega \) and \( \chi, \tau, \upsilon, \phi, \Sigma, \Xi, \Psi, \Omega \) are the vertices of the triangle.

The Little-known theorem by second-year student, John Smith.

For the following triangles, find the length of the base:

\[ \triangle ABC \]

\[ \triangle DEF \]

\[ \triangle GHI \]

\[ \triangle JKL \]

\[ \triangle MNP \]

\[ \triangle QRS \]

\[ \triangle TUV \]

\[ \triangle WXY \]

\[ \triangle ZAB \]

\[ \triangle CDE \]

\[ \triangle FGH \]

\[ \triangle IJK \]

\[ \triangle KLM \]

\[ \triangle NQ \]

\[ \triangle OPR \]

\[ \triangle QST \]

\[ \triangle UTV \]

\[ \triangle WUX \]

\[ \triangle XYZ \]

\[ \triangle ZAB \]

\[ \triangle CDE \]

\[ \triangle FGH \]

\[ \triangle IJK \]

\[ \triangle KLM \]

\[ \triangle NQ \]

\[ \triangle OPR \]

\[ \triangle QST \]

\[ \triangle UTV \]

\[ \triangle WUX \]

\[ \triangle XYZ \]
Vector spaces for a category, whose morphisms are given from functors. A linear functor $f : V \to W$ is grade-preserving if homogeneous elements have the same grade as their images, that is: a linear transformation $f : V \to W$ between vector spaces is grade-preserving if $f(V_0) \subseteq f(W_0)$ and $f(V_1) \subseteq f(W_1)$.

We can therefore define $\text{End}(V) = \text{Hom}(V, V)$.

When taking tensor products of vector superspaces, the $\mathbb{Z}_2$-grading is given additively:

$$[V \otimes W] = [V] + [W].$$
We have examined the Ray-Ban Equation before, as Chapter 3.

It is a special case of Plani 3.17, which has to do with V-modu...