The alternating central extension for the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$

Paul Terwilliger
The positive part $U^+_q$ of $U_q(\widehat{sl}_2)$ has a presentation with two generators $A, B$ that satisfy the cubic $q$-Serre relations.

Recently we introduced a type of element in $U^+_q$, said to be alternating. Each alternating element commutes with exactly one of

$$A, \quad B, \quad qAB - q^{-1}BA, \quad qBA - q^{-1}AB.$$ 

This gives four types of alternating elements; the elements of each type mutually commute.

We use these alternating elements to obtain a PBW basis for a certain central extension of $U^+_q$. 

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Recall the natural numbers \( \mathbb{N} = \{0, 1, 2, \ldots\} \) and integers \( \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} \).

Fix a field \( \mathbb{F} \).

Each vector space discussed is over \( \mathbb{F} \).

Each tensor product discussed is over \( \mathbb{F} \).

Each algebra discussed is associative, over \( \mathbb{F} \), and has a 1.
Let $A$ denote an algebra.

We will be discussing a type of basis for $A$, called a Poincaré-Birkhoff-Witt (or PBW) basis.

This consists of a subset $\Omega \subseteq A$ and a linear order $<$ on $\Omega$, such that the following is a linear basis for the vector space $A$:

$$a_1a_2 \cdots a_n \quad n \in \mathbb{N}, \quad a_1, a_2, \ldots, a_n \in \Omega,$$

$$a_1 \leq a_2 \leq \cdots \leq a_n.$$
Commutators and $q$-commutators

Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity.

Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad n \in \mathbb{Z}.$$ 

For elements $X, Y$ in any algebra, define their **commutator** and **$q$-commutator** by

$$[X, Y] = XY - YX, \quad [X, Y]_q = qXY - q^{-1}YX.$$ 

Note that

$$[X, [X, [X, Y]_q]_{q^{-1}}] = X^3Y - [3]_qX^2YX + [3]_qXYX^2 - YX^3.$$
The algebra $U_q^+$

**Definition**

Define the algebra $U_q^+$ by generators $A, B$ and relations

\[
[A, [A, [A, B]_q]_{q^{-1}}] = 0, \\
[B, [B, [B, A]_q]_{q^{-1}}] = 0.
\]

We call $U_q^+$ the **positive part of** $U_q(\widehat{sl}_2)$.

The above relations are called the **$q$-Serre relations**.
Why we care about $U_q^+$

We briefly explain why $U_q^+$ is of interest.

Let $V$ denote a finite-dimensional irreducible $U_q^+$-module on which $A, B$ are diagonalizable. Then:

- the eigenvalues of $A$ and $B$ on $V$ have the form
  
  \[
  A : \quad \{ a q^{d-2i} \}_{i=0}^d \quad 0 \neq a \in \mathbb{F}, \\
  B : \quad \{ b q^{d-2i} \}_{i=0}^d \quad 0 \neq b \in \mathbb{F}.
  \]

- For $0 \leq i \leq d$ let $V_i$ (resp. $V_i^*$) denote the eigenspace of $A$ (resp. $B$) for the eigenvalue $a q^{d-2i}$ (resp. $b q^{d-2i}$). Then
  
  \[
  B V_i \subseteq V_{i-1} + V_i + V_{i+1}, \\
  A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*,
  \]

  where $V_{-1} = 0 = V_{d+1}$ and $V_{-1}^* = 0 = V_{d+1}^*$. 

The alternating central extension for the positive part of $U_q(\hat{sl}_2)$
Consequently $A, B$ act on $V$ as a **tridiagonal pair**.

The topic of tridiagonal pairs is an active area of research, with links to

- combinatorics and graph theory (E. Bannai, T. Ito, W. Martin, S. Miklavic, K. Nomura, A. Pascasio, H. Tanaka);
- special functions and orthogonal polynomials (H. Alnajjar, B. Curtin, A. Grunbaum, E. Hanson, M. Ismail, J. H. Lee, R. Vidunas);
- quantum groups and representation theory (S. Bockting-Conrad, H. W. Huang, S. Kolb);
- mathematical physics (P. Baseilhac, S. Belliard, L. Vinet, A. Zhedanov)

We now return to $U_q^+$.  

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The alternating central extension for the positive part of $U_q(\hat{sl}_2)$
The alternating elements in $U_q^+$

Recently we introduced a type of element in $U_q^+$, said to be alternating.

Each alternating element commutes with exactly one of

$$A, \quad B, \quad qBA - q^{-1}AB, \quad qAB - q^{-1}BA.$$ 

This gives four types of alternating elements, denoted

$$\{W_{-k}\}_{k \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}, \quad \{G_{k+1}\}_{k \in \mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}.$$ 

The alternating elements of each type mutually commute.
The alternating elements in closed form

In order to describe the alternating elements in closed form, we use a $q$-shuffle algebra.

For this $q$-shuffle algebra, the underlying vector space is a free algebra on two generators.

This free algebra is described on the next slide.
Let $x, y$ denote noncommuting indeterminates.

Let $\mathbb{V}$ denote the free algebra with generators $x, y$.

By a **letter** in $\mathbb{V}$ we mean $x$ or $y$.

For $n \in \mathbb{N}$, a **word of length** $n$ in $\mathbb{V}$ is a product of letters $v_1 v_2 \cdots v_n$.

The vector space $\mathbb{V}$ has a linear basis consisting of its words.
We just defined the free algebra \( \mathbb{V} \).

There is another algebra structure on \( \mathbb{V} \), called the \textit{q-shuffle algebra}. This is due to M. Rosso 1995.

The \textit{q-shuffle product} is denoted by \( \star \).
The $q$-shuffle product on $\mathbb{V}$, cont.

For letters $u, v$ we have

$$u \star v = uv + vuq^{\langle u, v \rangle}$$

where

\[
\begin{array}{c|cc}
\langle , \rangle & x & y \\
\hline
x & 2 & -2 \\
y & -2 & 2
\end{array}
\]

So

$$x \star y = xy + q^{-2}yx,$$
$$y \star x = yx + q^{-2}xy,$$
$$x \star x = (1 + q^2)xx,$$
$$y \star y = (1 + q^2)yy.$$
For words $u, v$ in $\mathcal{V}$ we now describe $u \star v$.

Write $u = a_1 a_2 \cdots a_r$ and $v = b_1 b_2 \cdots b_s$.

To illustrate, assume $r = 2$ and $s = 2$.

We have

$$u \star v = a_1 a_2 b_1 b_2$$

$$+ a_1 b_1 a_2 b_2 q^{\langle a_2, b_1 \rangle}$$

$$+ a_1 b_1 b_2 a_2 q^{\langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle}$$

$$+ b_1 a_1 a_2 b_2 q^{\langle a_1, b_1 \rangle + \langle a_2, b_1 \rangle}$$

$$+ b_1 a_1 b_2 a_2 q^{\langle a_1, b_1 \rangle + \langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle}$$

$$+ b_1 b_2 a_1 a_2 q^{\langle a_1, b_1 \rangle + \langle a_1, b_2 \rangle + \langle a_2, b_1 \rangle + \langle a_2, b_2 \rangle}$$
The $q$-shuffle algebra

Theorem (Rosso 1995)

The $q$-shuffle product $\star$ turns the vector space $\nabla$ into an algebra.

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Theorem (Rosso 1995)

There exists an algebra homomorphism \( \natural \) from \( U_q^+ \) to the q-shuffle algebra \( \mathcal{V} \), that sends \( A \mapsto x \) and \( B \mapsto y \). The map \( \natural \) is injective.
The $q$-shuffle algebra, cont.

We can now easily describe the alternating elements in $U_q^+$. The map $\natural$ sends

\begin{align*}
W_0 & \mapsto x, & W_{-1} & \mapsto xyx, & W_{-2} & \mapsto xyxyx, & \ldots \\
W_1 & \mapsto y, & W_2 & \mapsto yxy, & W_3 & \mapsto yxyxy, & \ldots \\
G_1 & \mapsto yx, & G_2 & \mapsto yxyx, & G_3 & \mapsto yxyxyx, & \ldots \\
\tilde{G}_1 & \mapsto xy, & \tilde{G}_2 & \mapsto xyxy, & \tilde{G}_3 & \mapsto xyxyxy, & \ldots
\end{align*}
In the next three slides, we describe some relations that are satisfied by the alternating elements of $U_q^+$. For notational convenience define $G_0 = 1$ and $\tilde{G}_0 = 1$. 
Lemma (Type I relations)

For $k \in \mathbb{N}$ the following holds in $U_q^+$:

\[
[W_0, W_{k+1}] = [W_{-k}, W_1] = (1 - q^{-2})(\tilde{G}_{k+1} - G_{k+1}),
\]

\[
[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = (q - q^{-1})W_{-k-1},
\]

\[
[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = (q - q^{-1})W_{k+2}.
\]
Lemma (Type II relations)

For $k, \ell \in \mathbb{N}$ the following relations hold in $U_q^+$:

\[
\begin{align*}
\quad [W_{-k}, W_{-\ell}] &= 0, \quad [W_{k+1}, W_{\ell+1}] = 0, \\
\quad [W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] &= 0, \\
\quad [W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] &= 0, \\
\quad [W_{-k}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{-\ell}] &= 0, \\
\quad [W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] &= 0, \\
\quad [W_{k+1}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{\ell+1}] &= 0, \\
\quad [G_{k+1}, G_{\ell+1}] &= 0, \quad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0, \\
\quad [\tilde{G}_{k+1}, G_{\ell+1}] + [G_{k+1}, \tilde{G}_{\ell+1}] &= 0.
\end{align*}
\]
Lemma (Type III relations)

For $n \geq 1$ the following relations hold in $U^+_q$:

\[
\sum_{k=0}^{n} G_k \tilde{G}_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{-k} W_{n-k} q^{n-1-2k},
\]

\[
\sum_{k=0}^{n} G_k \tilde{G}_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{n-k} W_{-k} q^{n-1-2k},
\]

\[
\sum_{k=0}^{n} \tilde{G}_k G_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{n-k} W_{-k} q^{2k+1-n},
\]

\[
\sum_{k=0}^{n} \tilde{G}_k G_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{-k} W_{n-k} q^{2k+1-n}.
\]
It turns out that the relations of type I, II, III imply the $q$-Serre relations, which are the defining relations for $U_q^+$. Consequently we have the following.

**Lemma**

The algebra $U_q^+$ has a presentation by generators

$$\{ W_{-k} \}_{k \in \mathbb{N}}, \quad \{ W_{k+1} \}_{k \in \mathbb{N}}, \quad \{ G_{k+1} \}_{k \in \mathbb{N}}, \quad \{ \tilde{G}_{k+1} \}_{k \in \mathbb{N}}$$

and the relations of type I, II, III.
Obtaining the alternating elements from $A, B$

Using the relations of type I, II, III we can recursively express each alternating element as a polynomial in $A, B$.

The details are on the next slide.
Obtaining the alternating elements from $A, B$

Lemma

Using the equations below, the alternating elements in $U_q^+$ are recursively obtained from $A, B$ in the following order:

$$W_0, \quad W_1, \quad G_1, \quad \tilde{G}_1, \quad W_{-1}, \quad W_2, \quad G_2, \quad \tilde{G}_2, \quad \ldots$$

We have $W_0 = A$ and $W_1 = B$. For $n \geq 1$,

$$G_n = \frac{q \sum_{k=0}^{n-1} W_{-k} W_{n-k} q^{n-1-2k} - \sum_{k=1}^{n-1} G_k \tilde{G}_{n-k} q^{n-2k}}{q^n + q^{-n}} + \frac{W_n W_0 - W_0 \ast W_n}{(1 + q^{-2n})(1 - q^{-2})},$$

$$\tilde{G}_n = G_n + \frac{W_0 W_n - W_n W_0}{1 - q^{-2}}, \quad W_{-n} = \frac{q W_0 G_n - q^{-1} G_n W_0}{q - q^{-1}},$$

$$W_{n+1} = \frac{q G_n W_1 - q^{-1} W_1 G_n}{q - q^{-1}}.$$
It is tempting to guess that the alternating elements of $U_q^+$ form a PBW basis for $U_q^+$.

This guess is incorrect, but can be corrected as follows.
The alternating PBW basis for $U^+_q$

**Lemma (Terwilliger 2018)**

A PBW basis for $U^+_q$ is obtained by the elements

$$\{W_{-i}\}_{i \in \mathbb{N}}, \quad \{\tilde{G}_{j+1}\}_{j \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}$$

in any linear order $<$ that satisfies

$$W_{-i} < \tilde{G}_{j+1} < W_{k+1} \quad i, j, k \in \mathbb{N}.$$

The above PBW basis for $U^+_q$ will be called alternating.
The alternating PBW basis for $U_q^+$ is obtained from the set of alternating elements of $U_q^+$, by removing $\{G_{k+1}\}_{k \in \mathbb{N}}$.

This removal seems unnatural to us.

To fix the problem, we replace $U_q^+$ by a certain central extension of $U_q^+$, denoted $\mathcal{U}_q^+$. 
The algebra $\mathcal{U}_q^+$

**Definition**

We define the algebra $\mathcal{U}_q^+$ by generators

\[ \{ \mathcal{W}_k \}_{k \in \mathbb{N}}, \quad \{ \mathcal{W}_{k+1} \}_{k \in \mathbb{N}}, \quad \{ \mathcal{G}_{k+1} \}_{k \in \mathbb{N}}, \quad \{ \tilde{\mathcal{G}}_{k+1} \}_{k \in \mathbb{N}} \]

and the relations of type I, II from the previous slides. These generators are called **alternating**.

For notational convenience define $\mathcal{G}_0 = 1$ and $\tilde{\mathcal{G}}_0 = 1$. 
The algebras $U_q^+$ and $U_q^+$ are related as follows.

**Lemma**

There exists an algebra homomorphism $\gamma : U_q^+ \to U_q^+$ that sends

$$
\mathcal{W}_n \mapsto \mathcal{W}_n, \quad \mathcal{W}_{n+1} \mapsto \mathcal{W}_{n+1}, \quad \mathcal{G}_n \mapsto \mathcal{G}_n, \quad \tilde{\mathcal{G}}_n \mapsto \tilde{\mathcal{G}}_n
$$

for $n \in \mathbb{N}$. Moreover $\gamma$ is surjective.

Shortly we will describe the kernel of $\gamma$. 
It turns out that $U_q^+$ has a large center.

In order to describe this center, we bring in some polynomials.
A polynomial algebra

Definition

Let \( \{ z_n \}_{n=1}^{\infty} \) denote mutually commuting indeterminates. Let \( \mathbb{F}[z_1, z_2, \ldots] \) denote the algebra consisting of the polynomials in \( z_1, z_2, \ldots \) that have all coefficients in \( \mathbb{F} \). For notational convenience define \( z_0 = 1 \).
The algebras $\mathcal{U}_q^+$ and $\mathbb{F}[z_1, z_2, \ldots]$ are related as follows.

**Lemma**

There exists an algebra homomorphism $\eta : \mathcal{U}_q^+ \to \mathbb{F}[z_1, z_2, \ldots]$ that sends

\[
\mathcal{W}_-n \mapsto 0, \quad \mathcal{W}_{n+1} \mapsto 0, \quad \mathcal{G}_n \mapsto z_n, \quad \tilde{\mathcal{G}}_n \mapsto z_n
\]

for $n \in \mathbb{N}$. Moreover $\eta$ is surjective.

Shortly we will describe the kernel of $\eta$. 
We have indicated how $\mathcal{U}^+_q$ is related to $\mathcal{U}^+_q$ and $\mathbb{F}[z_1, z_2, \ldots]$.

Next we describe how $\mathcal{U}^+_q$ is related to the tensor product $\mathcal{U}^+_q \otimes \mathbb{F}[z_1, z_2, \ldots]$. 

The alternating central extension for the positive part of $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$
Theorem (Terwilliger 2019)

There exists an algebra isomorphism $\varphi : U_q^+ \rightarrow U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots]$ that sends

$\mathcal{W}_n \mapsto \sum_{k=0}^{n} W_{k-n} \otimes z_k$, \hspace{1cm} \mathcal{W}_{n+1} \mapsto \sum_{k=0}^{n} W_{n+1-k} \otimes z_k,$

$\mathcal{G}_n \mapsto \sum_{k=0}^{n} G_{n-k} \otimes z_k$, \hspace{1cm} \tilde{\mathcal{G}}_n \mapsto \sum_{k=0}^{n} \tilde{G}_{n-k} \otimes z_k$

for $n \in \mathbb{N}$. Moreover $\varphi$ sends

$\mathcal{W}_0 \mapsto W_0 \otimes 1$, \hspace{1cm} \mathcal{W}_1 \mapsto W_1 \otimes 1.$
We just gave an algebra isomorphism

$$\varphi : \mathcal{U}_q^+ \rightarrow \mathcal{U}_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots].$$

Over the next few slides, we describe how \( \varphi \) is related to \( \gamma \) and \( \eta \).
How $\varphi$ is related to $\gamma$

We now describe how $\varphi$ is related to $\gamma$.

There exists an algebra homomorphism $\theta : \mathbb{F}[z_1, z_2, \ldots] \to \mathbb{F}$ that sends $z_n \mapsto 0$ for $n \geq 1$.

The map $\theta$ is surjective.

Consequently the vector space $\mathbb{F}[z_1, z_2, \ldots]$ is the direct sum of $\mathbb{F}1$ and the kernel of $\theta$.

This kernel is the ideal of $\mathbb{F}[z_1, z_2, \ldots]$ generated by $\{z_n\}_{n=1}^{\infty}$. 

The alternating central extension for the positive part of $U_q(\hat{sl}_2)$
How $\varphi$ is related to $\gamma$, cont.

Lemma

The following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{U}_q^+ & \xrightarrow{\varphi} & \mathcal{U}_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots] \\
\gamma \downarrow & & \downarrow \text{id} \otimes \theta \\
\mathcal{U}_q^+ & \xrightarrow{x \mapsto x \otimes 1} & \mathcal{U}_q^+ \otimes \mathbb{F}
\end{array}
$$

$\text{id} = \text{identity map}$
Next we describe how $\varphi$ is related to $\eta$.

Since $U_q^+$ is generated by $A, B$ and the $q$-Serre relations are homogeneous, there exists an algebra homomorphism $\vartheta : U_q^+ \rightarrow \mathbb{F}$ that sends $A \mapsto 0$ and $B \mapsto 0$.

The map $\vartheta$ is surjective.

Consequently the vector space $U_q^+$ is the direct sum of $\mathbb{F}1$ and the kernel of $\vartheta$.

The kernel of $\vartheta$ is the two-sided ideal of $U_q^+$ generated by $A, B$. 
The map $\vartheta$ acts on the alternating elements of $U_q^+$ as follows.

The map $\vartheta$ sends

$$W_{-k} \mapsto 0, \quad W_{k+1} \mapsto 0, \quad G_{k+1} \mapsto 0, \quad \tilde{G}_{k+1} \mapsto 0$$

for $k \in \mathbb{N}$. 
Lemma

The following diagram commutes:

\[ \begin{array}{ccc}
\mathcal{U}_q^+ & \xrightarrow{\varphi} & \mathcal{U}_q^+ \otimes F[z_1, z_2, \ldots] \\
\eta & & \varphi \otimes \text{id} \\
F[z_1, z_2, \ldots] & \xrightarrow{x \mapsto 1 \otimes x} & F \otimes F[z_1, z_2, \ldots]
\end{array} \]
We have been discussing the algebra isomorphism

$$\varphi : \mathcal{U}_q^+ \rightarrow \mathcal{U}_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots].$$

Over the next few slides, we give some consequences of the isomorphism.
Definition
Let $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ denote the subalgebra of $U_q^+$ generated by $\mathcal{W}_0, \mathcal{W}_1$.

Lemma
There exists an algebra isomorphism $U_q^+ \rightarrow \langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ that sends $A \mapsto \mathcal{W}_0$ and $B \mapsto \mathcal{W}_1$. 
The center of $\mathcal{U}_q^+$

**Definition**

Let $\mathcal{Z}$ denote the center of $\mathcal{U}_q^+$.

It is known that the center of $\mathcal{U}_q^+$ is equal to $\mathbb{F}1$.

Consequently $\mathcal{Z}$ is the preimage of $\mathbb{F} \otimes \mathbb{F}[z_1, z_2, \ldots]$ under the isomorphism $\varphi$. 
The center of $\mathcal{U}_q^+$, cont.

Next we give a generating set for the center $\mathcal{Z}$.

**Lemma**

The subalgebra $\mathcal{Z}$ is generated by $\{Z_n^\vee\}_{n=1}^\infty$, where

$$Z_n^\vee = \sum_{k=0}^{n} G_k \tilde{G}_{n-k} q^{n-2k} - q \sum_{k=0}^{n-1} \mathcal{W}_k \mathcal{W}_{n-k} q^{n-1-2k}.$$
Next we describe how the isomorphism \( \varphi \) acts on \( \{ Z_n \}^{\infty}_{n=1} \).

**Lemma**

For \( n \geq 1 \) the isomorphism \( \varphi \) sends

\[
Z_n^\vee \mapsto 1 \otimes z_n^\vee ,
\]

where

\[
z_n^\vee = \sum_{k=0}^{n} z_k z_{n-k} q^{n-2k}.
\]
Lemma

The elements $\{z_n^\vee\}_{n=1}^\infty$ are algebraically independent. Moreover the elements $\{Z_n^\vee\}_{n=1}^\infty$ are algebraically independent.
How $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ and $\mathcal{Z}$ are related

The subalgebras $\langle \mathcal{W}_0, \mathcal{W}_1 \rangle$ and $\mathcal{Z}$ are related as follows.

**Lemma**

The multiplication map

$$\langle \mathcal{W}_0, \mathcal{W}_1 \rangle \otimes \mathcal{Z} \rightarrow \mathcal{U}_q^+$$

$w \otimes z \mapsto wz$

is an algebra isomorphism.
Using our results so far, we can recursively express each alternating generator for $\mathcal{U}_q^+$ in terms of $\mathcal{W}_0$, $\mathcal{W}_1$, $\{Z_n^\vee\}_{n=1}^\infty$.

The details are on the next slide.
The alternating generators in terms of $\mathcal{W}_0$, $\mathcal{W}_1$, $\{Z_n^\vee\}_{n=1}^\infty$

**Lemma**

Using the equations below, the alternating generators of $\mathcal{U}_q^+$ are recursively obtained from $\mathcal{W}_0, \mathcal{W}_1, \{Z_n^\vee\}_{n=1}^\infty$ in the following order:

$\mathcal{W}_0, \mathcal{W}_1, \mathcal{G}_1, \tilde{\mathcal{G}}_1, \mathcal{W}_{-1}, \mathcal{W}_2, \mathcal{G}_2, \tilde{\mathcal{G}}_2, \mathcal{W}_{-2}, \mathcal{W}_3, \ldots$

For $n \geq 1$,

$$
\mathcal{G}_n = \frac{Z_n^\vee + q \sum_{k=0}^{n-1} \mathcal{W}_{-k} \mathcal{W}_{n-k} q^{n-1-2k} - \sum_{k=1}^{n-1} \mathcal{G}_k \tilde{\mathcal{G}}_{n-k} q^{n-2k}}{q^n + q^{-n}} \left( \mathcal{W}_n \mathcal{W}_0 - \mathcal{W}_0 \mathcal{W}_n + \frac{\mathcal{W}_n \mathcal{W}_0 - \mathcal{W}_0 \mathcal{W}_n}{(1 + q^{-2n})(1 - q^{-2})} \right),
$$

$$
\tilde{\mathcal{G}}_n = \mathcal{G}_n + \frac{\mathcal{W}_0 \mathcal{W}_n - \mathcal{W}_n \mathcal{W}_0}{1 - q^{-2}},
$$

$$
\mathcal{W}_{n+1} = \frac{q \mathcal{G}_n \mathcal{W}_1 - q^{-1} \mathcal{W}_1 \mathcal{G}_n}{q - q^{-1}}.
$$
Recall the algebra homomorphism $\gamma : U_q^+ \to U_q^+$. 

**Lemma**

The following are the same:

(i) the kernel of $\gamma$;

(ii) the 2-sided ideal of $U_q^+$ generated by $\{Z_n^\vee\}_{n=1}^\infty$. 

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The vector space $U_q^+$ is the direct sum of the following:

(i) the kernel of $\gamma$;

(ii) the subalgebra $\langle W_0, W_1 \rangle$. 

Lemma
Recall the algebra homomorphism $\eta : U_q^+ \rightarrow \mathbb{F}[z_1, z_2, \ldots]$.

**Lemma**

The following are the same:

(i) the kernel of $\eta$;

(ii) the 2-sided ideal of $U_q^+$ generated by $\mathcal{W}_0$, $\mathcal{W}_1$. 
The kernel of $\eta$, cont.

**Lemma**

The vector space $\mathcal{U}_q^+$ is the direct sum of the following:

(i) the center $\mathcal{Z}$ of $\mathcal{U}_q^+$;

(ii) the kernel of $\eta$. 

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In the previous slides we described the algebra $\mathcal{U}_q^+$ from various points of view.

Using this description we were able to obtain the following result.
The alternating PBW basis for $\mathcal{U}_q^+$

**Theorem (Terwilliger 2019)**

A PBW basis for $\mathcal{U}_q^+$ is obtained by the elements

\[
\{\mathcal{W}_i\}_{i \in \mathbb{N}}, \quad \{\mathcal{G}_j\}_{j \in \mathbb{N}}, \quad \{\tilde{\mathcal{G}}_k\}_{k \in \mathbb{N}}, \quad \{\mathcal{W}_\ell\}_{\ell \in \mathbb{N}}
\]

in any linear order $<$ that satisfies

\[
\mathcal{W}_i < \mathcal{G}_{j+1} < \tilde{\mathcal{G}}_{k+1} < \mathcal{W}_{\ell+1} \quad \text{for} \quad i, j, k, \ell \in \mathbb{N}.
\]
In this talk, we recalled the algebra $U_q^+$ and its alternating elements

$$\{W_{-k}\}_{k \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}, \quad \{G_{k+1}\}_{k \in \mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}.$$ 

We showed how these elements satisfy some relations of type I–III.

We defined an algebra $U_q^+$ by generators

$$\{W_{-k}\}_{k \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}, \quad \{G_{k+1}\}_{k \in \mathbb{N}}, \quad \{\tilde{G}_{k+1}\}_{k \in \mathbb{N}}$$

and the relations of type I, II; these generators are called alternating.

We used an algebra isomorphism $\varphi : U_q^+ \rightarrow U_q^+ \otimes \mathbb{F}[z_1, z_2, \ldots]$ to described $U_q^+$ in various ways.

We showed how the alternating generators give a PBW basis for $U_q^+$.

THANK YOU FOR YOUR ATTENTION!

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