The Rahman polynomials and the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$

Plamen Iliev    Paul Terwilliger
The Rahman polynomials are a family of two-variable Krawtchouk polynomials.

We give an interpretation of these polynomials in terms of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$.

We will obtain the basic properties of the polynomials, such as the orthogonality and 7-term recurrence, from the properties of a certain finite-dimensional irreducible $\mathfrak{sl}_3(\mathbb{C})$-module $V$. 
Here is an outline of the talk.

- The definition of the Rahman polynomials
- Review of the orthogonality relations
- Two Cartan subalgebras $H$ and $\tilde{H}$ of $\mathfrak{sl}_3(\mathbb{C})$
- The antiautomorphism $\dagger$ of $\mathfrak{sl}_3(\mathbb{C})$
- The $\mathfrak{sl}_3(\mathbb{C})$-module $V$
- A bilinear form $\langle , \rangle$ on $V$
- The Rahman polynomials and $V$
In what follows \( \{p_i\}_{i=1}^4 \) denote complex numbers. They are essentially arbitrary, although certain combinations are forbidden in order to avoid dividing by zero.

Define

\[
\begin{align*}
t &= \frac{(p_1 + p_2)(p_1 + p_3)}{p_1(p_1 + p_2 + p_3 + p_4)}, \\
v &= \frac{(p_1 + p_2)(p_2 + p_4)}{p_2(p_1 + p_2 + p_3 + p_4)}, \\
u &= \frac{(p_1 + p_3)(p_3 + p_4)}{p_3(p_1 + p_2 + p_3 + p_4)}, \\
w &= \frac{(p_2 + p_4)(p_3 + p_4)}{p_4(p_1 + p_2 + p_3 + p_4)}.
\end{align*}
\]
Fix an integer $N \geq 0$ and let $a, b, c, d$ denote mutually commuting indeterminates.

Define

$$P(a, b, c, d) = \sum_{0 \leq i, j, k, \ell \atop i + j + k + \ell \leq N} \frac{(-a)^i (-b)^k (-c)^i (-d)^j}{i! j! k! \ell! (-N)^{i+j+k+\ell}} t^i u^j v^k w^\ell.$$ 

We are using the shifted factorial notation

$$\left(\alpha\right)_n = \alpha (\alpha + 1) \cdots (\alpha + n - 1) \quad n = 0, 1, 2, \ldots$$
For nonnegative integers $m, n$ whose sum is at most $N$ the corresponding **Rahman polynomial** is $P(m, n, c, d)$ in the variables $c, d$.

The corresponding **dual Rahman polynomial** is $P(a, b, m, n)$ in the variables $a, b$. 

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The Rahman polynomials and their duals satisfy an orthogonality relation which we now describe.

Define

\[ \nu = \frac{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)(p_3 + p_4)}{(p_1p_4 - p_2p_3)^2}. \]
The orthogonality relation for the Rahman polynomials

Define $\eta_0 = \nu^{-1}$ and
\[
\eta_1 = \frac{p_1 p_2(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)},
\]
\[
\eta_2 = \frac{p_3 p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_3)(p_2 + p_4)(p_3 + p_4)}.
\]

Define $\tilde{\eta}_0 = \nu^{-1}$ and
\[
\tilde{\eta}_1 = \frac{p_1 p_3(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_3 + p_4)},
\]
\[
\tilde{\eta}_2 = \frac{p_2 p_4(p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_2 + p_4)(p_3 + p_4)}.
\]

A short computation shows
\[
\eta_0 + \eta_1 + \eta_2 = 1, \quad \tilde{\eta}_0 + \tilde{\eta}_1 + \tilde{\eta}_2 = 1.
\]
The orthogonality relation for the Rahman polynomials

**Theorem**

(Rahman and Hoare 2007, Mizukawa and Tanaka 2004) Fix nonnegative integers \(s, t\) whose sum is at most \(N\), and nonnegative integers \(\sigma, \tau\) whose sum is at most \(N\). Then both

\[
\sum_{0 \leq i, j, k \atop i+j+k=N} P(j, k, s, t) P(j, k, \sigma, \tau) \tilde{\eta}_0^i \tilde{\eta}_1^j \tilde{\eta}_2^k \binom{N}{i, j, k} = \frac{\delta_{s\sigma} \delta_{t\tau}}{\tilde{k}_1^s \tilde{k}_2^t} \binom{N}{r, s, t}^{-1},
\]

\[
\sum_{0 \leq i, j, k \atop i+j+k=N} P(s, t, j, k) P(\sigma, \tau, j, k) \eta_0^i \eta_1^j \eta_2^k \binom{N}{i, j, k} = \frac{\delta_{s\sigma} \delta_{t\tau}}{k_1^s k_2^t} \binom{N}{r, s, t}^{-1},
\]

where \(r = N - s - t\) and \(k_i = \nu \tilde{\eta}_i, \tilde{k}_i = \nu \eta_i\).
The connection to $\mathfrak{sl}_3(\mathbb{C})$

We now relate the Rahman polynomials to $\mathfrak{sl}_3(\mathbb{C})$.

For $0 \leq i, j \leq 2$ let $e_{ij}$ denote the matrix in $\text{Mat}_3(\mathbb{C})$ that has $(i, j)$-entry 1 and all other entries 0.

We will consider two Cartan subalgebras of $\mathfrak{sl}_3(\mathbb{C})$, denoted $H$ and $\tilde{H}$.

The subalgebra $H$ consists of the diagonal matrices in $\mathfrak{sl}_3(\mathbb{C})$.

Define

$$\varphi = \text{diag}(-1/3, 2/3, -1/3), \quad \phi = \text{diag}(-1/3, -1/3, 2/3).$$

Then $\varphi, \phi$ form a basis for $H$.

We now describe $\tilde{H}$. 
The Cartan subalgebra $\tilde{H}$

Define

$$\tilde{H} = RHR^{-1},$$

where

$$R = \begin{pmatrix}
\frac{p_2 p_3 - p_1 p_4}{(p_1 + p_3)(p_2 + p_4)} & \frac{p_2 p_3 - p_1 p_4}{(p_1 + p_3)(p_2 + p_4)} & \frac{p_2 p_3 - p_1 p_4}{(p_1 + p_3)(p_2 + p_4)} \\
\frac{p_1 p_3}{(p_1 + p_3)(p_2 + p_4)} & -\frac{p_3}{p_1 + p_3} & \frac{p_1}{p_1 + p_3} \\
\frac{p_2 p_4}{(p_2 + p_4)(p_2 p_3 - p_1 p_4)} & \frac{p_4}{p_2 + p_4} & -\frac{p_2}{p_2 + p_4}
\end{pmatrix}.$$ 

$\tilde{H}$ is a Cartan subalgebra of $\mathfrak{sl}_3(\mathbb{C})$.

It turns out that $H, \tilde{H}$ generate $\mathfrak{sl}_3(\mathbb{C})$. 

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Define

\[ \tilde{\varphi} = R\varphi R^{-1}, \quad \tilde{\phi} = R\phi R^{-1}. \]

Note that \( \tilde{\varphi}, \tilde{\phi} \) is a basis for \( \tilde{\mathcal{H}} \).

For \( 0 \leq i, j \leq 2 \) define \( \tilde{e}_{ij} = R e_{ij} R^{-1} \).
The antiautomorphism † of $\mathfrak{sl}_3(\mathbb{C})$

The Cartan subalgebras $H, ˜H$ are related via a certain antiautomorphism † of $\mathfrak{sl}_3(\mathbb{C})$.

By definition

$$\beta^\dagger = \tilde{W} \beta^t \tilde{W}^{-1} \quad \forall \beta \in \mathfrak{sl}_3(\mathbb{C}),$$

where

$$\tilde{W} = \text{diag}(\tilde{\eta}_0, \tilde{\eta}_1, \tilde{\eta}_2).$$
We have

\[
\begin{array}{c|ccc|ccc|c}
\beta & e_{01} & e_{12} & e_{02} & e_{10} & e_{21} & e_{20} & \varphi \\
\hline
\beta^{\dagger} & e_{10}\tilde{\eta}_1/\tilde{\eta}_0 & e_{21}\tilde{\eta}_2/\tilde{\eta}_1 & e_{20}\tilde{\eta}_2/\tilde{\eta}_0 & e_{01}\tilde{\eta}_0/\tilde{\eta}_1 & e_{12}\tilde{\eta}_1/\tilde{\eta}_2 & e_{02}\tilde{\eta}_0/\tilde{\eta}_2 & \varphi \\
\end{array}
\]

and

\[
\begin{array}{c|ccc|ccc|c}
\beta & \tilde{e}_{01} & \tilde{e}_{12} & \tilde{e}_{02} & \tilde{e}_{10} & \tilde{e}_{21} & \tilde{e}_{20} & \varphi \\
\hline
\beta^{\dagger} & \tilde{e}_{10}\eta_1/\eta_0 & \tilde{e}_{21}\eta_2/\eta_1 & \tilde{e}_{20}\eta_2/\eta_0 & \tilde{e}_{01}\eta_0/\eta_1 & \tilde{e}_{12}\eta_1/\eta_2 & \tilde{e}_{02}\eta_0/\eta_2 & \varphi \\
\end{array}
\]

Note that \(\dagger\) fixes each element of \(H\) and each element of \(\tilde{H}\).
We now define a certain \( \mathfrak{sl}_3(\mathbb{C}) \)-module.

Let \( x, y, z \) denote mutually commuting indeterminates. Let \( \mathbb{C}[x, y, z] \) denote the \( \mathbb{C} \)-algebra consisting of the polynomials in \( x, y, z \) that have all coefficients in \( \mathbb{C} \). We abbreviate \( A = \mathbb{C}[x, y, z] \).

The space \( A \) is an \( \mathfrak{sl}_3(\mathbb{C}) \)-module on which each element of \( \mathfrak{sl}_3(\mathbb{C}) \) acts as a derivation and

<table>
<thead>
<tr>
<th>( \xi )</th>
<th>( e_{01}.\xi )</th>
<th>( e_{12}.\xi )</th>
<th>( e_{02}.\xi )</th>
<th>( e_{10}.\xi )</th>
<th>( e_{21}.\xi )</th>
<th>( e_{20}.\xi )</th>
<th>( \varphi.\xi )</th>
<th>( \phi.\xi )</th>
</tr>
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<td>0</td>
<td>0</td>
<td>( y )</td>
<td>0</td>
<td>( z )</td>
<td>(-x/3)</td>
<td>(-x/3)</td>
</tr>
<tr>
<td>( y )</td>
<td>( x )</td>
<td>0</td>
<td>0</td>
<td>( 0 )</td>
<td>( z )</td>
<td>0</td>
<td>( 2y/3 )</td>
<td>(-y/3)</td>
</tr>
<tr>
<td>( z )</td>
<td>0</td>
<td>( y )</td>
<td>( x )</td>
<td>( 0 )</td>
<td>0</td>
<td>( 0 )</td>
<td>(-z/3)</td>
<td>( 2z/3 )</td>
</tr>
</tbody>
</table>
An $\mathfrak{sl}_3(\mathbb{C})$-module

Let $V$ denote the subspace of $A$ consisting of the homogeneous polynomials that have total degree $N$.

The following is a basis for $V$:

$$\begin{align*}
&x^r y^s z^t \quad r \geq 0, \quad s \geq 0, \quad t \geq 0, \quad r + s + t = N.
\end{align*}$$

Call this the **monomial basis**. The action of $\mathfrak{sl}_3(\mathbb{C})$ on this basis is described as follows.

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$\xi e_{01}$</th>
<th>$\xi e_{12}$</th>
<th>$\xi e_{02}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^r y^s z^t$</td>
<td>$s x^{r+1} y^{s-1} z^t$</td>
<td>$t x^r y^{s+1} z^{t-1}$</td>
<td>$t x^{r+1} y^s z^{t-1}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$\xi e_{10}$</th>
<th>$\xi e_{21}$</th>
<th>$\xi e_{20}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^r y^s z^t$</td>
<td>$r x^{r-1} y^{s+1} z^t$</td>
<td>$s x^r y^{s-1} z^{t+1}$</td>
<td>$r x^{r-1} y^s z^{t+1}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$\varphi \xi$</th>
<th>$\phi \xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^r y^s z^t$</td>
<td>$(s - N/3)x^r y^s z^t$</td>
<td>$(t - N/3)x^r y^s z^t$</td>
</tr>
</tbody>
</table>

The space $V$ is an $\mathfrak{sl}_3(\mathbb{C})$-submodule of $A$ which turns out to be irreducible.
We now consider the $H$-weight space decomposition of $V$.

Let $\mathbb{I}$ denote the set consisting of the 3-tuples of nonnegative integers whose sum is $N$.

For $\lambda = (r, s, t) \in \mathbb{I}$ let $V_\lambda$ denote the subspace of $V$ spanned by $x^r y^s z^t$. Then

$$V = \sum_{\lambda \in \mathbb{I}} V_\lambda$$

(direct sum).

This is the $H$-weight space decomposition of $V$.

By construction $\dim(V_\lambda) = 1$ for all $\lambda \in \mathbb{I}$. 

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We now consider the $\tilde{H}$-weight space decomposition of $V$.

To describe this decomposition we make a change of variables.

Recall the matrix $R$ and define

\[
\tilde{x} = R_{00}x + R_{10}y + R_{20}z,
\]

\[
\tilde{y} = R_{01}x + R_{11}y + R_{21}z,
\]

\[
\tilde{z} = R_{02}x + R_{12}y + R_{22}z.
\]

Thus $R$ is the transition matrix from $x, y, z$ to $\tilde{x}, \tilde{y}, \tilde{z}$. 
The action of $\mathfrak{sl}_3(\mathbb{C})$ on $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ is described as follows.

<table>
<thead>
<tr>
<th>$\xi$</th>
<th>$\tilde{e}_{01}.\xi$</th>
<th>$\tilde{e}_{12}.\xi$</th>
<th>$\tilde{e}_{02}.\xi$</th>
<th>$\tilde{e}_{10}.\xi$</th>
<th>$\tilde{e}_{21}.\xi$</th>
<th>$\tilde{e}_{20}.\xi$</th>
<th>$\tilde{\phi}.\xi$</th>
<th>$\tilde{\phi}.\xi$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>0</td>
<td>$\tilde{y}$</td>
<td>0</td>
<td>$\tilde{z}$</td>
<td>$-\tilde{x}/3$</td>
<td>$-\tilde{x}/3$</td>
</tr>
<tr>
<td>$\tilde{y}$</td>
<td>$\tilde{x}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\tilde{z}$</td>
<td>0</td>
<td>$2\tilde{y}/3$</td>
<td>$-\tilde{y}/3$</td>
</tr>
<tr>
<td>$\tilde{z}$</td>
<td>0</td>
<td>$\tilde{y}$</td>
<td>$\tilde{x}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-\tilde{z}/3$</td>
<td>$2\tilde{z}/3$</td>
</tr>
</tbody>
</table>
The $\tilde{H}$-weight space decomposition of $V$

The following is a basis for $V$.

$$\tilde{x}^r \tilde{y}^s \tilde{z}^t \quad r \geq 0, \quad s \geq 0, \quad t \geq 0, \quad r + s + t = N.$$  

Call this the **dual monomial basis**. The action of $\mathfrak{sl}_3(\mathbb{C})$ on this basis is described as follows.

$$
\begin{array}{c|ccc}
\xi & \tilde{e}_{01}.\xi & \tilde{e}_{12}.\xi & \tilde{e}_{02}.\xi \\
\tilde{x}^r \tilde{y}^s \tilde{z}^t & s\tilde{x}^{r+1} \tilde{y}^{s-1} \tilde{z}^t & t\tilde{x}^r \tilde{y}^{s+1} \tilde{z}^{t-1} & t\tilde{x}^{r+1} \tilde{y}^{s} \tilde{z}^{t-1} \\
\xi & \tilde{e}_{10}.\xi & \tilde{e}_{21}.\xi & \tilde{e}_{20}.\xi \\
\tilde{x}^r \tilde{y}^s \tilde{z}^t & r\tilde{x}^{r-1} \tilde{y}^{s+1} \tilde{z}^t & s\tilde{x}^r \tilde{y}^{s-1} \tilde{z}^{t+1} & r\tilde{x}^{r-1} \tilde{y}^s \tilde{z}^{t+1} \\
\xi & \tilde{\phi}.\xi & \tilde{\phi}.\xi \\
\tilde{x}^r \tilde{y}^s \tilde{z}^t & (s - N/3)\tilde{x}^r \tilde{y}^{s} \tilde{z}^t & (t - N/3)\tilde{x}^r \tilde{y}^s \tilde{z}^t
\end{array}
$$
For each $\lambda = (r, s, t) \in \mathbb{I}$ let $\tilde{V}_\lambda$ denote the subspace of $V$ spanned by $\tilde{x}^r\tilde{y}^s\tilde{z}^t$.

Observe that

$$V = \sum_{\lambda \in \mathbb{I}} \tilde{V}_\lambda \quad \text{(direct sum).}$$

This is the $\tilde{H}$-weight space decomposition of $V$.

By construction $\dim(\tilde{V}_\lambda) = 1$ for all $\lambda \in \mathbb{I}$.  

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The Rahman polynomials and the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$
The action of $H, \tilde{H}$ on each other's weight spaces

We comment on how $H$ and $\tilde{H}$ act on the weight spaces of the other one.

A pair of elements $(r, s, t)$ and $(r', s', t')$ in $\mathbb{I}$ will be called adjacent whenever $(r - r', s - s', t - t')$ is a permutation of $(1, -1, 0)$.

A generic element in $\mathbb{I}$ is adjacent to six elements of $\mathbb{I}$.

$H$ and $\tilde{H}$ act on each other's weight spaces as follows.

For all $\lambda \in \mathbb{I}$,

$$\tilde{H}V_\lambda \subseteq V_\lambda + \sum_{\substack{\mu \in \mathbb{I} \\ \mu \text{ adj } \lambda}} V_\mu, \quad H\tilde{V}_\lambda \subseteq \tilde{V}_\lambda + \sum_{\substack{\mu \in \mathbb{I} \\ \mu \text{ adj } \lambda}} \tilde{V}_\mu.$$
We now introduce a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $V$.

As we will see, both

\[
\langle V_\lambda, V_\mu \rangle = 0 \quad \text{if} \quad \lambda \neq \mu, \quad \lambda, \mu \in \mathbb{I},
\]

\[
\langle \tilde{V}_\lambda, \tilde{V}_\mu \rangle = 0 \quad \text{if} \quad \lambda \neq \mu, \quad \lambda, \mu \in \mathbb{I}.
\]
We define $\langle , \rangle$ as follows. With respect to $\langle , \rangle$ the monomial basis is orthogonal and

$$\|x^ry^sz^t\|^2 = \frac{r!s!t!}{\tilde{\eta}_0\tilde{\eta}_1\tilde{\eta}_2} \vartheta^N \quad r \geq 0, \quad s \geq 0, \quad t \geq 0, \quad r + s + t = N.$$ 

The form $\langle , \rangle$ is symmetric and nondegenerate. Moreover

$$\langle \beta\xi, \zeta \rangle = \langle \xi, \beta^\dagger\zeta \rangle \quad \forall \beta \in \mathfrak{sl}_3(\mathbb{C}), \quad \forall \xi, \zeta \in V.$$ 

Consequently with respect to $\langle , \rangle$ the dual monomial basis is orthogonal. Also

$$\|\tilde{x}^r \tilde{y}^s \tilde{z}^t\|^2 = \frac{r!s!t!}{\eta_0\eta_1\eta_2} \vartheta^N \quad r \geq 0, \quad s \geq 0, \quad t \geq 0, \quad r + s + t = N.$$
We now state our main results.

Earlier we defined the monomial basis and dual monomial basis for the \( \mathfrak{sl}_3(\mathbb{C}) \)-module \( V \). These bases are related as follows.

**Theorem**

For nonnegative integers \( s, t \) whose sum is at most \( N \), both

\[
P(s, t, \bar{\varphi} + N/3 \mathbf{I}, \tilde{\varphi} + N/3 \mathbf{I})x^N = x^r y^s z^t, \\
P(\varphi + N/3 \mathbf{I}, \phi + N/3 \mathbf{I}, s, t)\bar{x}^N = \bar{x}^r \bar{y}^s \bar{z}^t,
\]

where \( r = N - s - t \).
The Rahman polynomials as transition matrix entries

Recall the monomial basis and dual monomial bases for the $\mathfrak{sl}_3(\mathbb{C})$-module $V$.

The next result shows that for each transition matrix the entries are described by Rahman polynomials and their duals.

**Theorem**

For nonnegative integers $\rho, \sigma, \tau$ whose sum is $N$, both

\[
\tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau = N! \nu^N \sum_{0 \leq r, s, t \atop r+s+t=N} P(s, t, \sigma, \tau) \frac{\tilde{\eta}_0 \tilde{\eta}_1 \tilde{\eta}_2}{r!s!t!} \frac{x^r y^s z^t}{\tilde{\vartheta}^N}.
\]

\[
x^\rho y^\sigma z^\tau = N! \nu^N \sum_{0 \leq r, s, t \atop r+s+t=N} P(\sigma, \tau, s, t) \frac{\eta_0^r \eta_1^s \eta_2^t}{r!s!t!} \frac{x^r y^s z^t}{\tilde{\vartheta}^N}.
\]
Referring to the $\mathfrak{sl}_3(\mathbb{C})$-module $V$, the next result shows that for a vector in the monomial basis and a vector in the dual monomial basis, their inner product is described by a Rahman polynomial.

**Theorem**

For a vector $x^r y^s z^t$ from the monomial basis and a vector $\tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau$ from the dual monomial basis,

$$\langle x^r y^s z^t, \tilde{x}^\rho \tilde{y}^\sigma \tilde{z}^\tau \rangle = N! \nu^N P(s, t, \sigma, \tau).$$
At the beginning of the talk we displayed some orthogonality relations for the Rahman polynomials.

These relations can be recovered from our analysis of $\mathcal{V}$.

The relations reflect the fact that both the monomial basis and dual monomial basis are orthogonal with respect to $\langle , \rangle$. 
We now show that the Rahman polynomials satisfy some 7-term recurrence relations.

The significance of the 7 is that $7 - 1 = 6$ is the number of roots in the root system $A_2$ associated with $\mathfrak{sl}_3(\mathbb{C})$. 
In the next result we display two 7-term recurrence relations satisfied by the Rahman polynomials, along with similar recurrences satisfied by the dual polynomials.

**Theorem**

*Fix nonnegative integers* \( s, t \) *whose sum is at most* \( N \), *and nonnegative integers* \( \sigma, \tau \) *whose sum is at most* \( N \). *Then the following hold.*
Two 7-term recurrences

**Theorem**

(i) \((s - N/3)P(s, t, \sigma, \tau)\) is a weighted sum with the following terms and coefficients:

<table>
<thead>
<tr>
<th>term</th>
<th>coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P(s, t, \sigma - 1, \tau))</td>
<td>(\sigma \frac{p_3(p_1p_4 - p_2p_3)}{(p_1+p_2)(p_1+p_3)(p_3+p_4)})</td>
</tr>
<tr>
<td>(P(s, t, \sigma, \tau - 1))</td>
<td>(\tau \frac{p_1(p_2p_3 - p_1p_4)}{(p_1+p_2)(p_1+p_3)(p_3+p_4)})</td>
</tr>
<tr>
<td>(P(s, t, \sigma + 1, \tau))</td>
<td>(\rho \frac{p_1p_2}{p_1p_3p_4(p_1+p_2+p_3+p_4)})</td>
</tr>
<tr>
<td>(P(s, t, \sigma + 1, \tau - 1))</td>
<td>(\tau \frac{p_3p_4}{p_1p_3p_4(p_1+p_2+p_3+p_4)})</td>
</tr>
<tr>
<td>(P(s, t, \sigma, \tau + 1))</td>
<td>(\rho \frac{-p_3p_4}{(p_1+p_3)(p_3+p_4)})</td>
</tr>
<tr>
<td>(P(s, t, \sigma - 1, \tau + 1))</td>
<td>((\sigma - N/3)\left(\frac{p_2p_3}{(p_1+p_2)(p_1+p_3)} - \frac{p_1p_3(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_1+p_3)(p_3+p_4)}\right))</td>
</tr>
<tr>
<td>(P(s, t, \sigma, \tau))</td>
<td>((\sigma - N/3)\left(\frac{p_1p_4}{(p_1+p_3)(p_3+p_4)} - \frac{p_1p_3(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_1+p_3)(p_3+p_4)}\right)) + ((\tau - N/3)\left(\frac{p_1p_4}{(p_1+p_3)(p_3+p_4)} - \frac{p_1p_3(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_1+p_3)(p_3+p_4)}\right))</td>
</tr>
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</table>

In the above table \(r = N - s - t\) and \(\rho = N - \sigma - \tau\).
Two 7-term recurrences, cont.

Theorem

(ii) \((t - N/3)P(s, t, \sigma, \tau)\) is a weighted sum with the following terms and coefficients:

<table>
<thead>
<tr>
<th>term</th>
<th>coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P(s, t, \sigma - 1, \tau))</td>
<td>(\sigma \frac{p_4(p_2p_3 - p_1p_4)}{(p_1+p_2)(p_2+p_4)(p_3+p_4)})</td>
</tr>
<tr>
<td>(P(s, t, \sigma, \tau - 1))</td>
<td>(\tau \frac{p_2(p_1p_4 - p_2p_3)}{(p_1+p_2)(p_2+p_4)(p_3+p_4)})</td>
</tr>
<tr>
<td>(P(s, t, \sigma + 1, \tau))</td>
<td>(\rho \frac{p_1p_2}{p_1p_2p_4(p_1+p_2+p_3+p_4)})</td>
</tr>
<tr>
<td>(P(s, t, \sigma + 1, \tau - 1))</td>
<td>(\tau \frac{-p_1p_2}{(p_1+p_2)(p_2+p_4)})</td>
</tr>
<tr>
<td>(P(s, t, \sigma, \tau + 1))</td>
<td>(\rho \frac{-p_3p_4}{p_2p_3p_4(p_1+p_2+p_3+p_4)})</td>
</tr>
<tr>
<td>(P(s, t, \sigma - 1, \tau + 1))</td>
<td>((\sigma - N/3)\left(\frac{p_1p_4}{(p_1+p_2)(p_2+p_4)} - \frac{p_2p_4(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_2+p_4)(p_3+p_4)}\right))</td>
</tr>
<tr>
<td>(P(s, t, \sigma, \tau))</td>
<td>((\tau - N/3)\left(\frac{p_2p_3}{(p_2+p_4)(p_3+p_4)} - \frac{p_2p_4(p_1+p_2+p_3+p_4)}{(p_1+p_2)(p_2+p_4)(p_3+p_4)}\right))</td>
</tr>
</tbody>
</table>

In the above table \(r = N - s - t\) and \(\rho = N - \sigma - \tau\).
(iii) \((\sigma - N/3)P(s, t, \sigma, \tau)\) is a weighted sum with the following terms and coefficients:

<table>
<thead>
<tr>
<th>term</th>
<th>coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P(s - 1, t, \sigma, \tau))</td>
<td>(s \frac{p_2(p_1 p_4 - p_2 p_3)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)})</td>
</tr>
<tr>
<td>(P(s, t - 1, \sigma, \tau))</td>
<td>(t \frac{p_1(p_2 p_3 - p_1 p_4)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)})</td>
</tr>
<tr>
<td>(P(s + 1, t, \sigma, \tau))</td>
<td>(r \frac{p_1 p_2 p_3 (p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)})</td>
</tr>
<tr>
<td>(P(s + 1, t - 1, \sigma, \tau))</td>
<td>(t \frac{p_1 p_2 p_4 (p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)})</td>
</tr>
<tr>
<td>(P(s, t + 1, \sigma, \tau))</td>
<td>(r \frac{-p_2 p_4}{(p_1 + p_2)(p_2 + p_4)})</td>
</tr>
<tr>
<td>(P(s - 1, t + 1, \sigma, \tau))</td>
<td>(s \frac{p_2 p_3}{(p_1 + p_2)(p_1 + p_3)} - \frac{p_1 p_2 (p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)})</td>
</tr>
<tr>
<td>(P(s, t, \sigma, \tau))</td>
<td>((s - N/3) \left( \frac{p_2 p_3}{(p_1 + p_2)(p_1 + p_3)} - \frac{p_1 p_2 (p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)} \right)) + ((t - N/3) \left( \frac{p_1 p_4}{(p_1 + p_2)(p_2 + p_4)} - \frac{p_1 p_2 (p_1 + p_2 + p_3 + p_4)}{(p_1 + p_2)(p_1 + p_3)(p_2 + p_4)} \right))</td>
</tr>
</tbody>
</table>

In the above table \(r = N - s - t\) and \(\rho = N - \sigma - \tau\).
Two 7-term recurrences, cont.

Theorem

(iv) \((\tau - N/3)P(s, t, \sigma, \tau)\) is a weighted sum with the following terms and coefficients:

<table>
<thead>
<tr>
<th>term</th>
<th>coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P(s - 1, t, \sigma, \tau))</td>
<td>(s \frac{p_4(p_2p_3-p_1p_4)}{(p_1+p_3)(p_3+p_4)(p_2+p_4)})</td>
</tr>
<tr>
<td>(P(s, t - 1, \sigma, \tau))</td>
<td>(t \frac{p_3(p_1p_4-p_2p_3)}{(p_1+p_3)(p_3+p_4)(p_2+p_4)})</td>
</tr>
<tr>
<td>(P(s + 1, t, \sigma, \tau))</td>
<td>(r \frac{p_1p_3}{p_1p_3p_4(p_1+p_2+p_3+p_4)})</td>
</tr>
<tr>
<td>(P(s + 1, t - 1, \sigma, \tau))</td>
<td>(t \frac{p_1p_3}{(p_1+p_3)(p_3+p_4)})</td>
</tr>
<tr>
<td>(P(s, t + 1, \sigma, \tau))</td>
<td>(\frac{p_2p_3}{p_2p_3p_4(p_1+p_2+p_3+p_4)})</td>
</tr>
<tr>
<td>(P(s - 1, t + 1, \sigma, \tau))</td>
<td>(-\frac{p_2p_4}{(p_2+p_4)(p_3+p_4)})</td>
</tr>
<tr>
<td>(P(s, t, \sigma, \tau))</td>
<td>((s - N/3)\left(\frac{p_1p_4}{(p_1+p_3)(p_3+p_4)} - \frac{p_3p_4(p_1+p_2+p_3+p_4)}{(p_1+p_3)(p_3+p_4)(p_2+p_4)}\right))</td>
</tr>
<tr>
<td>(P(s, t, \sigma, \tau))</td>
<td>((t - N/3)\left(\frac{p_2p_3}{(p_2+p_4)(p_3+p_4)} - \frac{p_3p_4(p_1+p_2+p_3+p_4)}{(p_1+p_3)(p_3+p_4)(p_2+p_4)}\right))</td>
</tr>
</tbody>
</table>

In the above table \(r = N - s - t\) and \(\rho = N - \sigma - \tau\).
We interpreted the Rahman polynomials in terms of the Lie algebra \( \mathfrak{sl}_3(\mathbb{C}) \).

Using the parameters of the polynomials we defined two Cartan subalgebras for \( \mathfrak{sl}_3(\mathbb{C}) \), denoted \( H \) and \( \tilde{H} \).

We displayed an antiautomorphism \( \dagger \) of \( \mathfrak{sl}_3(\mathbb{C}) \) that fixes each element of \( H \) and each element of \( \tilde{H} \).

We considered a certain finite-dimensional irreducible \( \mathfrak{sl}_3(\mathbb{C}) \)-module \( V \) consisting of homogeneous polynomials in three variables.

We displayed a nondegenerate symmetric bilinear form \( \langle , \rangle \) on \( V \) such that \( \langle \beta \xi, \zeta \rangle = \langle \xi, \beta^\dagger \zeta \rangle \) for all \( \beta \in \mathfrak{sl}_3(\mathbb{C}) \) and \( \xi, \zeta \in V \).
Summary, cont.

We displayed two bases for \( V \); one diagonalizes \( H \) and the other diagonalizes \( \tilde{H} \). Both bases are orthogonal with respect to \( \langle \cdot, \cdot \rangle \).

We showed that when \( \langle \cdot, \cdot \rangle \) is applied to a vector in each basis, the result is a trivial factor times a Rahman polynomial evaluated at an appropriate argument.

Thus for both transition matrices between the bases each entry is described by a Rahman polynomial. From these results we recover the previously known orthogonality relation for the Rahman polynomials.

We also obtained two seven-term recurrence relations satisfied by the Rahman polynomials, along with the corresponding relations satisfied by the dual polynomials.

Thank you for your attention!

THE END