Leonard pairs, spin models, and distance-regular graphs

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The work of Caughman, Curtin, and Nomura shows that for a distance-regular graph $\Gamma$ affording a spin model, the irreducible modules for the subconstituent algebra $\mathcal{T}$ take a certain form.

We show that the converse is true: whenever all the irreducible $\mathcal{T}$-modules take this form, then $\Gamma$ affords a spin model.

We explicitly construct this spin model when $\Gamma$ has $q$-Racah type. This is joint work with Kazumasa Nomura.
We are the first to admit: we have not discovered any new spin model to date.

What we have shown, is that a new spin model would result from the discovery of a new distance-regular graph with the right sort of irreducible T-modules.
Let $X$ denote a nonempty finite set.

Let $V$ denote the vector space over $\mathbb{C}$ consisting of the column vectors whose entries are indexed by $X$.

For $y \in X$ define the vector $\hat{y} \in V$ that has $y$-entry 1 and all other entries 0.

Note that $\{\hat{y}\}_{y \in X}$ form a basis for $V$.

For a real number $\alpha > 0$ let $\alpha^{1/2}$ denote the positive square root of $\alpha$. 

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A matrix $W \in \text{Mat}_X(\mathbb{C})$ is said to be type II whenever $W$ is symmetric with all entries nonzero and

$$
\sum_{y \in X} \frac{W(a, y)}{W(b, y)} = |X| \delta_{a,b} \quad (a, b \in X).
$$
Next we recall the **Nomura algebra** of a type II matrix.

**Definition**

Assume \( W \in \text{Mat}_X(\mathbb{C}) \) is type II. For \( b, c \in X \) define

\[
u_{b,c} = \sum_{y \in X} \frac{W(b, y)}{W(c, y)} \hat{y}.
\]

Further define

\[
N(W) = \{ B \in \text{Mat}_X(\mathbb{C}) \mid B \text{ is symmetric}, \ B\nu_{b,c} \in \mathbb{C}\nu_{b,c} \text{ for all } b, c \in X \}.
\]
Lemma (Nomura 1997)

Assume $W \in \text{Mat}_X(\mathbb{C})$ is type II.

Then $N(W)$ is a commutative subalgebra of $\text{Mat}_X(\mathbb{C})$ that contains the all 1’s matrix $J$ and is closed under the Hadamard product.

We call $N(W)$ the **Nomura algebra** of $W$. 
A matrix $W \in \text{Mat}_X(\mathbb{C})$ is called a **spin model** whenever $W$ is type II and

$$
\sum_{y \in X} \frac{W(a, y)W(b, y)}{W(c, y)} = |X|^{1/2} \frac{W(a, b)}{W(a, c)W(b, c)}
$$

for all $a, b, c \in X$. 

**Definition**
Lemma (Nomura 1997)

Assume $W \in \text{Mat}_X(\mathbb{C})$ is a spin model. Then $W \in N(W)$.
Hadamard matrices

Definition
A matrix $H \in \text{Mat}_X(\mathbb{C})$ is called **Hadamard** whenever every entry is $\pm 1$ and $HH^t = |X|I$.

Example
The matrix

$$H = \begin{pmatrix}
1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 \\
-1 & -1 & -1 & 1
\end{pmatrix}$$

is Hadamard.
A symmetric Hadamard matrix is type II.

More generally, for $W \in \text{Mat}_X(\mathbb{C})$ and $0 \neq \alpha \in \mathbb{C}$ the following are equivalent:

(i) $W$ is type II with all entries $\pm \alpha$;

(ii) there exists a symmetric Hadamard matrix $H$ such that $W = \alpha H$.

**Definition**

A type II matrix $W \in \text{Mat}_X(\mathbb{C})$ is said to have **Hadamard type** whenever $W$ is a scalar multiple of a symmetric Hadamard matrix.
We briefly consider spin models of Hadamard type.

**Example**

Recall our example $H$ of a Hadamard matrix. Then $W = \sqrt{-1} H$ is a spin model of Hadamard type.

Spin models of Hadamard type sometimes cause technical problems, so occasionally we will assume that a spin model under discussion does not have Hadamard type.
Let $\Gamma$ denote a distance-regular graph, with vertex set $X$ and diameter $D \geq 3$.

Let $M$ denote the Bose-Mesner algebra of $\Gamma$.

Assume that $M$ contains a spin model $W$.

**Definition**

We say that $\Gamma$ **affords** $W$ whenever $W \in M \subseteq N(W)$.
When $\Gamma$ affords a spin model

Until further notice, assume that the spin model $W$ is afforded by $\Gamma$.

We now consider the consequences.
The graph $\Gamma$ is formally self-dual

Lemma (Curtin+Nomura 1999)

There exists an ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents of $M$ with respect to which $\Gamma$ is formally self-dual.

For this ordering the intersection numbers and Krein parameters satisfy

$$p_{ij}^h = q_{ij}^h \quad (0 \leq h, i, j \leq D).$$
Corollary

The graph $\Gamma$ is $Q$-polynomial with respect to the ordering $\{E_i\}_{i=0}^D$. 
Since \( \{E_i\}_{i=0}^D \) is a basis for \( M \) and \( W \) is an invertible element in \( M \), there exist nonzero scalars \( f, \{\tau_i\}_{i=0}^D \) in \( \mathbb{C} \) such that \( \tau_0 = 1 \) and

\[
W = f \sum_{i=0}^D \tau_i E_i.
\]

Note that

\[
W^{-1} = f^{-1} \sum_{i=0}^D \tau_i^{-1} E_i.
\]
Recall that the distance-matrices \( \{ A_i \}_{i=0}^D \) form a basis for \( M \).

**Lemma (Curtin 1999)**

We have

\[
W = |X|^{1/2} f^{-1} \sum_{i=0}^{D} \tau_i^{-1} A_i,
\]

\[
W^{-1} = |X|^{-3/2} f \sum_{i=0}^{D} \tau_i A_i.
\]
The parameter $f$

**Lemma**

The scalar $f$ satisfies

$$f^{-2} = |X|^{-3/2} \sum_{i=0}^{D} k_i \tau_i,$$

where $\{k_i\}_{i=0}^{D}$ are the valencies of $\Gamma$.

We call the above equation the **standard normalization**.
We now bring in the dual Bose-Mesner algebra.

Until further notice, fix a vertex $x \in X$.

For $0 \leq i \leq D$ let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ that has $(y, y)$-entry 1 if $\partial(x, y) = i$ and 0 if $\partial(x, y) \neq i$ ($y \in X$). By construction,

$$E_i^*E_j^* = \delta_{i,j}E_i^*, \quad (0 \leq i, j \leq D), \quad \sum_{i=0}^{D} E_i^* = I.$$

Consequently $\{E_i^*\}_{i=0}^{D}$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\text{Mat}_X(\mathbb{C})$, called the **dual Bose-Mesner algebra** of $\Gamma$ with respect to $x$. 
The matrix $W^*$

Define $W^* = W^*(x)$ by

$$W^* = f \sum_{i=0}^{D} \tau_i E_i^*.$$  

Note that

$$(W^*)^{-1} = f^{-1} \sum_{i=0}^{D} \tau_i^{-1} E_i^*.$$
Next we recall the dual distance-matrices.

For $0 \leq i \leq D$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ whose $(y, y)$-entry is the $(x, y)$-entry of $|X|E_i$ ($y \in X$). We have $A_0^* = I$ and

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^* \quad (0 \leq i, j \leq D).$$

The matrices $\{A_i^*\}_{i=0}^{D}$ form a basis for $M^*$. We call $\{A_i^*\}_{i=0}^{D}$ the dual distance-matrices of $\Gamma$ with respect to $x$. 

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Lemma (Curtin 1999)

We have

\[ W^* = |X|^{1/2} f^{-1} \sum_{i=0}^{D} \tau_i^{-1} A_i^*, \]

\[ (W^*)^{-1} = |X|^{-3/2} f \sum_{i=0}^{D} \tau_i A_i^*. \]
How $W$, $W^*$ are related

Next we consider how $W$, $W^*$ are related.

**Lemma (Munemasa 1994, Caughman and Wolff 2005)**

*We have*

\[ WA_1^* W^{-1} = (W^*)^{-1} A_1 W^*, \]

\[ WW^* W = W^* WW^*. \]
We now bring in the subconstituent algebra.

Let $T = T(x)$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $M$ and $M^*$. 

We call $T$ the subconstituent algebra of $\Gamma$ with respect to $x$. 
We now describe the irreducible T-modules.

**Lemma (Curtin 1999)**

*Each irreducible T-module is thin, provided that W is not of Hadamard type.*

**Lemma (Curtin and Nomura 2004)**

*Let U denote a thin irreducible T-module. Then the endpoint of U is equal to the dual-endpoint of U.*
In order to further describe the irreducible $T$-modules, we recall a concept from linear algebra.

**Definition**

Let $V$ denote a vector space over $\mathbb{C}$ with finite positive dimension. By a **Leonard pair** on $V$ we mean an ordered pair of $\mathbb{C}$-linear maps $A : V \to V$ and $A^* : V \to V$ that satisfy the following (i), (ii).

(i) There exists a basis for $V$ with respect to which the matrix representing $A$ is irreducible tridiagonal and the matrix representing $A^*$ is diagonal.

(ii) There exists a basis for $V$ with respect to which the matrix representing $A^*$ is irreducible tridiagonal and the matrix representing $A$ is diagonal.
Spin Leonard pairs

Definition

Let $A, A^*$ denote a Leonard pair on $V$. A **balanced Boltzmann pair** for $A, A^*$ is an ordered pair of invertible linear maps $W : V \to V$ and $W^* : V \to V$ such that

(i) $WA = AW$;
(ii) $W^* A^* = A^* W^*$;
(iii) $WA^* W^{-1} = (W^*)^{-1} AW^*$;
(iv) $WW^* W = W^* WW^*$.

Lemma

The Leonard pair $A, A^*$ is said to have **spin** whenever there exists a balanced Boltzmann pair for $A, A^*$.
Curtin (2007) classified up to isomorphism the spin Leonard pairs and described their Boltzmann pairs.

We now return our attention to the graph $\Gamma$.

**Lemma (Caughman and Wolff 2005)**

*The pair $A_1, A_1^*$ acts on each irreducible $T$-module $U$ as a spin Leonard pair, and $W, W^*$ acts on $U$ as a balanced Boltzmann pair for this Leonard pair.*
We have been discussing a distance-regular graph $\Gamma$ that affords a spin model $W$.

We showed that the existence of $W$ implies that the irreducible $T$-modules take a certain form.

We now reverse the logical direction.

We show that whenever the irreducible $T$-modules take this form, then $\Gamma$ affords a spin model $W$. 
A condition on $\Gamma$

Let $\Gamma$ denote a distance-regular graph with vertex set $X$ and diameter $D \geq 3$.

**Assumption**

Assume that $\Gamma$ is formally self-dual with respect to the ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents.
A condition on the irreducible T-modules

Definition

Let \( f, \{\tau_i\}_{i=0}^{D} \) denote nonzero scalars in \( \mathbb{C} \) such that \( \tau_0 = 1 \). Define

\[
W = f \sum_{i=0}^{D} \tau_i E_i.
\]

For \( x \in X \) define

\[
W^*(x) = f \sum_{i=0}^{D} \tau_i E_i^*(x).
\]
The first main theorem

Assume that for all \( x \in X \) and all irreducible \( T(x) \)-modules \( U \),

(i) \( U \) is thin;

(ii) \( U \) has the same endpoint and dual-endpoint;

(iii) the pair \( A_1, A_1^*(x) \) acts on \( U \) as a spin Leonard pair, and \( W, W^*(x) \) acts on \( U \) as a balanced Boltzmann pair for this spin Leonard pair;

(iv) \( f \) satisfies the standard normalization equation.

Then \( W \) is a spin model afforded by \( \Gamma \).
Next we make the previous theorem more explicit, under the assumption that $\Gamma$ has $q$-Racah type.

**Assumption**

Assume that $\Gamma$ is formally self-dual with respect to the ordering $\{E_i\}_{i=0}^D$ of the primitive idempotents.

Fix nonzero scalars $a, q \in \mathbb{C}$ such that

\[
q^{2i} \neq 1 \quad (1 \leq i \leq D),
\]

\[
a^2 q^{2i} \neq 1 \quad (1 - D \leq i \leq D - 1),
\]

\[
a^3 q^{2i-D-1} \neq 1 \quad (1 \leq i \leq D).
\]
An assumption on the eigenvalues

For $0 \leq i \leq D$ let $\theta_i$ denote the eigenvalue of the adjacency matrix $A_1$ for $E_i$.

Assumption

Assume that

$$\theta_i = \alpha(aq^{2i-D} + a^{-1}q^{D-2i}) + \beta \quad (0 \leq i \leq D),$$

where

$$\alpha = \frac{(aq^{2-D} - a^{-1}q^{D-2})(a + q^{D-1})}{q^{D-1}(q^{-1} - q)(aq - a^{-1}q^{-1})(a - q^{1-D})},$$

$$\beta = \frac{q(a + a^{-1})(a + q^{D-1})(aq^{2-D} - a^{-1}q^{D-2})}{(q - q^{-1})(a - q^{1-D})(aq - a^{-1}q^{-1})}.$$
Assumption

Assume that the intersection numbers of $\Gamma$ satisfy

\[
b_i = \frac{\alpha(q^{i-D} - q^{D-i})(aq^{i-D} - a^{-1}q^{D-i})(a^3 - q^{D-2i-1})}{a(aq^{2i-D} - a^{-1}q^{D-2i})(a + q^{D-2i-1})},
\]

\[
c_i = \frac{\alpha a(q^i - q^{-i})(aq^i - a^{-1}q^{-i})(a^{-1} - q^{D-2i+1})}{(aq^{2i-D} - a^{-1}q^{D-2i})(a + q^{D-2i+1})},
\]

for $1 \leq i \leq D - 1$ and

\[
b_0 = \frac{\alpha(q^{-D} - q^D)(a^3 - q^{D-1})}{a(a + q^{D-1})},
\]

\[
c_D = \frac{\alpha(q^{-D} - q^D)(a - q^{D-1})}{q^{D-1}(a + q^{1-D})}.
\]
An assumption on the irreducible $T$-modules

**Assumption**

Assume that for all $x \in X$ and all irreducible $T(x)$-modules $U$,

(i) $U$ is thin;

(ii) $U$ has the same endpoint and dual-endpoint (called $r$);

(iii) the intersection numbers $\{c_i(U)\}_{i=1}^{d}$, $\{b_i(U)\}_{i=0}^{d-1}$ satisfy

\[
\begin{align*}
    b_i(U) &= \frac{\alpha(q^{-d} - q^d)(a^3 - q^{3D-2d-6r-2i-1})}{aq^{D-d-2r}(aq^{2r+i-D} - a^{-1}q^{D-2r-2i})(a + q^{D-2r-2i-1})}, \\
    c_i(U) &= \frac{\alpha a(q^i - q^{-i})(aq^{d+2r+i-D} - a^{-1}q^{D-d-2r-i})(a^{-1} - q^{2D-D+2r-2i+1})}{q^{d-D+2r}(aq^{2r+i-D} - a^{-1}q^{D-2r-2i})(a + q^{D-2r-2i+1})},
\end{align*}
\]

for $1 \leq i \leq d - 1$ and

\[
\begin{align*}
    b_0(U) &= \frac{\alpha(q^{-d} - q^d)(a^3 - q^{3D-2d-6r-1})}{aq^{D-d-2r}(a + q^{D-2r-1})}, \\
    c_d(U) &= \frac{\alpha(q^{-d} - q^d)(a - q^{D-2r-1})}{q^{d-1}(a + q^{D-2d-2r+1})}.
\end{align*}
\]
Constructing a spin model $W$

**Theorem**

Define scalars $\{\tau_i\}_{i=0}^{D}$ in $\mathbb{C}$ by

$$
\tau_i = (-1)^i a^{-i} q^{i(D-i)} \quad (0 \leq i \leq D).
$$

Define $f \in \mathbb{C}$ such that

$$
f^2 = \frac{|X|^{3/2} (aq^{1-D}; q^2)_D}{(a^{-2}; q^2)_D}.
$$

Then the matrix

$$
W = f \sum_{i=0}^{D} \tau_i E_i
$$

is a spin model afforded by $\Gamma$. 
In this talk we considered a distance-regular graph $\Gamma$.

We first assumed that $\Gamma$ affords a spin model, and showed that the irreducible modules for the subconstituent algebra $T$ take a certain form.

We then reversed the logical direction. We assumed that all the irreducible $T$-modules take this form, and showed that $\Gamma$ affords a spin model.

We explicitly constructed this spin model when $\Gamma$ has $q$-Racah type.

THANK YOU FOR YOUR ATTENTION!