The alternating PBW basis for the positive part of $U_q(\hat{\mathfrak{sl}}_2)$

Paul Terwilliger
The positive part $U_q^+$ of $U_q(\widehat{\mathfrak{sl}}_2)$ has a presentation with two generators $A, B$ that satisfy the cubic $q$-Serre relations.

We introduce a PBW basis for $U_q^+$, said to be alternating.

Each element of this PBW basis commutes with exactly one of $A, B, qAB - q^{-1}BA$.

This gives three types of PBW basis elements; the elements of each type mutually commute.

We interpret the alternating PBW basis in terms of a $q$-shuffle algebra associated with affine $\mathfrak{sl}_2$.

We show how the alternating PBW basis is related to the PBW basis for $U_q^+$ found by Damiani in 1993.
Our discovery of the alternating PBW basis was inspired by the work of mathematical physicists P. Baseilhac, K. Koizumi, K. Shigechi concerning boundary integrable systems with hidden symmetries.

We were led to the alternating PBW basis while trying to understand their work.

Simply put, our discovery would not have occurred without their inspiration.
Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

Fix a field $\mathbb{F}$.

Each vector space discussed is over $\mathbb{F}$.

Each algebra discussed is associative, over $\mathbb{F}$, and has a 1.
PBW bases

Let $\mathcal{A}$ denote an algebra.

We will be discussing a type of basis for $\mathcal{A}$, called a Poincaré-Birkhoff-Witt (or PBW) basis.

This consists of a subset $\Omega \subseteq \mathcal{A}$ and a linear order $<$ on $\Omega$, such that the following is a linear basis for the vector space $\mathcal{A}$:

$$a_1 a_2 \cdots a_n \quad n \in \mathbb{N}, \quad a_1, a_2, \ldots, a_n \in \Omega,$$

$$a_1 \leq a_2 \leq \cdots \leq a_n.$$
Commutators and $q$-commutators

Fix a nonzero $q \in \mathbb{F}$ that is not a root of unity.

Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n \in \mathbb{Z}.$$  

For elements $X, Y$ in any algebra, define their **commutator** and **$q$-commutator** by

$$[X, Y] = XY - YX, \quad [X, Y]_q = qXY - q^{-1} YX.$$  

Note that

$$[X, [X, [X, Y]_q]_{q^{-1}}] = X^3 Y - [3]_q X^2 YX + [3]_q XYX^2 - YX^3.$$  

The alternating PBW basis for the positive part of $U_q(\hat{sl}_2)$
The algebra $U_q^+$

Definition

Define the algebra $U_q^+$ by generators $A, B$ and relations

\[
[A, [A, [A, B]_q]_{q^{-1}}] = 0, \\
[B, [B, [B, A]_q]_{q^{-1}}] = 0.
\]

We call $U_q^+$ the **positive part of** $U_q(\hat{sl}_2)$.

The above relations are called the **$q$-Serre relations**.
Why we care about $U_q^+$

We briefly explain why $U_q^+$ is of interest.

Let $V$ denote a finite-dimensional irreducible $U_q^+$-module on which $A, B$ are diagonalizable. Then:

- the eigenvalues of $A$ and $B$ on $V$ have the form

  
  $A : \left\{ aq^{d-2i} \right\}_{i=0}^{d} \quad 0 \neq a \in \mathbb{F},$

  $B : \left\{ bq^{d-2i} \right\}_{i=0}^{d} \quad 0 \neq b \in \mathbb{F}.$

- For $0 \leq i \leq d$ let $V_i$ (resp. $V_i^*$) denote the eigenspace of $A$ (resp. $B$) for the eigenvalue $aq^{d-2i}$ (resp. $bq^{d-2i}$). Then

  
  $BV_i \subseteq V_{i-1} + V_i + V_{i+1},$

  $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*,$

where $V_{-1} = 0 = V_{d+1}$ and $V_{-1}^* = 0 = V_{d+1}^*.$
Consequently $A, B$ act on $V$ as a **tridiagonal pair**.

The topic of tridiagonal pairs is an active area of research, with links to

- combinatorics and graph theory (E. Bannai, T. Ito, W. Martin, S. Miklavec, K. Nomura, A. Pascasio, H. Tanaka);
- special functions and orthogonal polynomials (H. Alnajjar, B. Curtin, A. Grunbaum, E. Hanson, M. Ismail, J. H. Lee, R. Vidunas);
- quantum groups and representation theory (S. Bockting-Conrad, H. W. Huang, S. Kolb);
- mathematical physics (P. Baseilhac, S. Belliard, L. Vinet, A. Zhedanov)

We now return to $U_q^+$. 

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The alternating PBW basis for the positive part of $U_q(\hat{sl}_2)$
An $\mathbb{N}^2$-grading for $U_q^+$

The algebra $U_q^+$ has a grading that we now describe.

Note that the $q$-Serre relations are homogeneous in both $A$ and $B$.

Therefore the algebra $U_q^+$ has an $\mathbb{N}^2$-grading such that $A$ and $B$ are homogeneous, with degrees $(1, 0)$ and $(0, 1)$ respectively.

For $(i, j) \in \mathbb{N}^2$ let $d_{i,j}$ denote the dimension of the $(i, j)$-homogeneous component of $U_q^+$.

These dimensions are described by the generating function

$$
\sum_{(i,j) \in \mathbb{N}^2} d_{i,j} \lambda^i \mu^j = \prod_{\ell=1}^{\infty} \frac{1}{1 - \lambda^\ell \mu^{\ell-1}} \frac{1}{1 - \lambda^\ell \mu} \frac{1}{1 - \lambda^{\ell-1} \mu^\ell}.
$$
For $0 \leq i, j \leq 6$ the dimension $d_{i,j}$ is given in the $(i, j)$-entry of the matrix below:

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 3 & 3 & 3 & 3 & 3 \\
1 & 3 & 6 & 8 & 9 & 9 & 9 \\
1 & 3 & 8 & 14 & 19 & 21 & 22 \\
1 & 3 & 9 & 19 & 32 & 42 & 48 \\
1 & 3 & 9 & 21 & 42 & 66 & 87 \\
1 & 3 & 9 & 22 & 48 & 87 & 134 \\
\end{pmatrix}
$$
In 1993, Damiani obtained a PBW basis for $U_q^+$, involving some elements

$$\{E_{n\delta+\alpha_0}\}_{n=0}^\infty, \quad \{E_{n\delta+\alpha_1}\}_{n=0}^\infty, \quad \{E_{n\delta}\}_{n=1}^\infty.$$  \hfill (1)

These elements are recursively defined as follows:

$$E_{\alpha_0} = A, \quad E_{\alpha_1} = B, \quad E_\delta = q^{-2}BA - AB,$$

and for $n \geq 1$,

$$E_{n\delta+\alpha_0} = \frac{[E_\delta, E_{(n-1)\delta+\alpha_0}]}{q + q^{-1}}, \quad E_{n\delta+\alpha_1} = \frac{[E_{(n-1)\delta+\alpha_1}, E_\delta]}{q + q^{-1}},$$

$$E_{n\delta} = q^{-2}E_{(n-1)\delta+\alpha_1}A - AE_{(n-1)\delta+\alpha_1}.$$
The elements (1) are homogeneous with degrees shown below:

<table>
<thead>
<tr>
<th>element</th>
<th>degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{n\delta+\alpha_0}$</td>
<td>$(n + 1, n)$</td>
</tr>
<tr>
<td>$E_{n\delta+\alpha_1}$</td>
<td>$(n, n + 1)$</td>
</tr>
<tr>
<td>$E_{n\delta}$</td>
<td>$(n, n)$</td>
</tr>
</tbody>
</table>
Theorem (Damiani 1993)

A PBW basis for $U_q^+$ is given by the elements (1) in linear order

$$E_{\alpha_0} < E_{\delta+\alpha_0} < E_{2\delta+\alpha_0} < \cdots$$

$$\cdots < E_{\delta} < E_{2\delta} < E_{3\delta} < \cdots$$

$$\cdots < E_{2\delta+\alpha_1} < E_{\delta+\alpha_1} < E_{\alpha_1}.$$

Moreover the elements $\{E_{n\delta}\}_{n=1}^{\infty}$ mutually commute.
The Damiani PBW basis elements are defined recursively.

Next we describe these elements in closed form, using a $q$-shuffle algebra.

For this $q$-shuffle algebra, the underlying vector space is a free algebra on two generators.

This free algebra is described on the next slide.
Let $x, y$ denote noncommuting indeterminates.

Let $\mathbb{V}$ denote the free algebra with generators $x, y$.

By a letter in $\mathbb{V}$ we mean $x$ or $y$.

For $n \in \mathbb{N}$, a word of length $n$ in $\mathbb{V}$ is a product of letters $v_1 v_2 \cdots v_n$.

The vector space $\mathbb{V}$ has a linear basis consisting of its words; this basis is called standard.
We just defined the free algebra $\mathcal{V}$.

There is another algebra structure on $\mathcal{V}$, called the $q$-shuffle algebra. This is due to M. Rosso 1995.

The $q$-shuffle product will be denoted by $\star$. 

The alternating PBW basis for the positive part of $U_q(\hat{sl}_2)$...
For letters $u, v$ we have

$$u \star v = uv + vuq^{\langle u, v \rangle}$$

where

$$\langle , \rangle \begin{array}{c|cc} x & y \\ \hline x & 2 & -2 \\ y & -2 & 2 \end{array}$$

So

$$x \star y = xy + q^{-2}yx, \quad y \star x = yx + q^{-2}xy, \quad x \star x = (1 + q^2)xx, \quad y \star y = (1 + q^2)yy.$$
The \( q \)-shuffle product on \( \mathbb{V} \), cont.

For words \( u, v \) in \( \mathbb{V} \) we now describe \( u \star v \).

Write \( u = a_1 a_2 \cdots a_r \) and \( v = b_1 b_2 \cdots b_s \).

To illustrate, assume \( r = 2 \) and \( s = 2 \).

We have

\[
\begin{align*}
    u \star v &= a_1 a_2 b_1 b_2 \\
               &+ a_1 b_1 a_2 b_2 q^{\langle a_2, b_1 \rangle} \\
               &+ a_1 b_1 b_2 a_2 q^{\langle a_2, b_1 \rangle} + q^{\langle a_2, b_2 \rangle} \\
               &+ b_1 a_1 a_2 b_2 q^{\langle a_1, b_1 \rangle} + q^{\langle a_2, b_1 \rangle} \\
               &+ b_1 a_1 b_2 a_2 q^{\langle a_1, b_1 \rangle} + q^{\langle a_2, b_1 \rangle} + q^{\langle a_2, b_2 \rangle} \\
               &+ b_1 b_2 a_1 a_2 q^{\langle a_1, b_1 \rangle} + q^{\langle a_1, b_2 \rangle} + q^{\langle a_2, b_1 \rangle} + q^{\langle a_2, b_2 \rangle}
\end{align*}
\]
Theorem (Rosso 1995)

The q-shuffle product $\star$ turns the vector space $\mathcal{V}$ into an algebra.
The algebra $U$

**Definition**

Let $U$ denote the subalgebra of the $q$-shuffle algebra $V$ generated by $x, y$.

The algebra $U$ is described as follows. We have

\[
\begin{align*}
  x \star x \star x \star x \star y - [3]_q x \star x \star x \star x \star x + [3]_q x \star y \star x \star x \star x - y \star x \star x \star x = 0, \\
y \star y \star y \star x - [3]_q y \star y \star x \star y + [3]_q y \star x \star y \star y \star y - x \star y \star y \star y \star y = 0.
\end{align*}
\]

So in the $q$-shuffle algebra $V$ the elements $x, y$ satisfy the $q$-Serre relations.

Consequently there exists an algebra homomorphism $\flat$ from $U_q^+$ to the $q$-shuffle algebra $V$, that sends $A \mapsto x$ and $B \mapsto y$.

The map $\flat$ has image $U$ by construction.
How $U_q^+$ is related to $U$.

Theorem (Rosso, 1995)

The map $\hat{\eta}: U_q^+ \to U$ is an algebra isomorphism.

Next we describe how the map $\hat{\eta}$ acts on the Damiani PBW basis for $U_q^+$.
The Catalan words in $\mathcal{V}$

Give each letter $x, y$ a weight:

$$\overline{x} = 1, \quad \overline{y} = -1.$$ 

A word $v_1 v_2 \cdots v_n$ in $\mathcal{V}$ is **Catalan** whenever $\overline{v_1} + \overline{v_2} + \cdots + \overline{v_i}$ is nonnegative for $1 \leq i \leq n - 1$ and zero for $i = n$. In this case $n$ is even.

**Example**

For $0 \leq n \leq 3$ we give the Catalan words of length $2n$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Catalan words of length $2n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>$xy$</td>
</tr>
<tr>
<td>2</td>
<td>$xyxy, ; xyy$</td>
</tr>
<tr>
<td>3</td>
<td>$xyxyxy, ; xyyxyx, ; xyxxyy, ; xyyxyy, ; xxyyyy$</td>
</tr>
</tbody>
</table>
The Damiani PBW basis in closed form

**Definition**

For \( n \in \mathbb{N} \) define

\[
C_n = \sum \nu_1 \nu_2 \cdots \nu_{2n} \prod_{i=1}^{2n} \left( 1 + \nu_1 + \nu_2 + \cdots + \nu_{2i} \right),
\]

where the sum is over all the Catalan words \( \nu_1 \nu_2 \cdots \nu_{2n} \) in \( \mathbb{V} \) that have length \( 2n \).

**Example**

We have

\[
C_0 = 1, \quad C_1 = [2]_q xy, \quad C_2 = [2]_q^2 xyxy + [3]_q [2]_q^2 xyy, \\
C_3 = [2]_q^3 xyxyxy + [3]_q [2]_q^3 xyyxy + [3]_q [2]_q^3 xyxyx \\
\quad + [3]_q^2 [2]_q^3 xyy + [4]_q [3]_q^2 [2]_q^2 xxxx.
\]
Theorem (Terwilliger 2018)

The map $\natural$ sends

$$E_{n\delta + \alpha_0} \mapsto q^{-2n} (q - q^{-1})^{2n} \times C_n,$$
$$E_{n\delta + \alpha_1} \mapsto q^{-2n} (q - q^{-1})^{2n} C_n y$$

for $n \geq 0$, and

$$E_{n\delta} \mapsto -q^{-2n} (q - q^{-1})^{2n-1} C_n$$

for $n \geq 1$. 
The \( \{ C_n \}_{n=1}^{\infty} \) mutually commute.

We mentioned earlier that \( \{ E_{n\delta} \}_{n=1}^{\infty} \) mutually commute.

**Corollary**

For \( i, j \in \mathbb{N} \),

\[
C_i \star C_j = C_j \star C_i.
\]
We just described the Damiani PBW basis for $U_q^+$.

In the coming slides, we will obtain another PBW basis for $U_q^+$, said to be alternating.

This PBW basis has the following features.
For the alternating PBW basis,

- each element commutes with exactly one of $A$, $B$, $[A, B]_q$;
- the elements that commute with $A$ mutually commute;
- the elements that commute with $B$ mutually commute;
- the elements that commute with $[A, B]_q$ mutually commute.

From now on, we identify $U^+_q$ with $U$, via the isomorphism $\natural$. 

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The alternating words in $\mathbb{V}$

**Definition**

A word $v_1 v_2 \cdots v_n$ in $\mathbb{V}$ is called **alternating** whenever $n \geq 1$ and $v_{i-1} \neq v_i$ for $2 \leq i \leq n$. Thus an alternating word has the form $\cdots xyxy \cdots$. 

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The alternating PBW basis for the positive part of $U_q(\widehat{sl}_2)$
The alternating words, cont.

**Definition**

We name the alternating words as follows:

\[ W_0 = x, \quad W_{-1} = xyx, \quad W_{-2} = xyxyx, \quad \ldots \]
\[ W_1 = y, \quad W_2 = yxy, \quad W_3 = yxyxy, \quad \ldots \]
\[ G_1 = yx, \quad G_2 = yxyx, \quad G_3 = yxyxyx, \quad \ldots \]
\[ \tilde{G}_1 = xy, \quad \tilde{G}_2 = xyxy, \quad \tilde{G}_3 = xyxyxy, \quad \ldots \]

For notational convenience define \( G_0 = 1 \) and \( \tilde{G}_0 = 1 \).
The alternating words, cont.

For $k \in \mathbb{N}$,

<table>
<thead>
<tr>
<th>name</th>
<th>description</th>
<th>$x$-degree</th>
<th>$y$-degree</th>
<th>length</th>
</tr>
</thead>
<tbody>
<tr>
<td>$W_{-k}$</td>
<td>xyxy $\cdots$ x</td>
<td>$k + 1$</td>
<td>$k$</td>
<td>$2k + 1$</td>
</tr>
<tr>
<td>$W_{k+1}$</td>
<td>xxyx $\cdots$ y</td>
<td>$k$</td>
<td>$k + 1$</td>
<td>$2k + 1$</td>
</tr>
<tr>
<td>$G_k$</td>
<td>xxyx $\cdots$ x</td>
<td>$k$</td>
<td>$k$</td>
<td>$2k$</td>
</tr>
<tr>
<td>$\tilde{G}_k$</td>
<td>xxyy $\cdots$ y</td>
<td>$k$</td>
<td>$k$</td>
<td>$2k$</td>
</tr>
</tbody>
</table>
We are going to show that \( U \) contains the alternating words.

As a warmup, consider the alternating words \( xy \) and \( yx \).

One checks

\[
xy = q \frac{qx \ast y - q^{-1}y \ast x}{q^2 - q^{-2}}, \quad yx = q \frac{qy \ast x - q^{-1}x \ast y}{q^2 - q^{-2}}.
\]

Therefore \( U \) contains \( xy \) and \( yx \).
Over the next four slides, we display many relations satisfied by the alternating words.

These relations will imply that $U$ contains the alternating words.
Relations for the alternating words, I

Lemma

For $k \in \mathbb{N}$ the following holds in the $q$-shuffle algebra $\mathbb{V}$:

\[
[W_0, W_{k+1}] = [W_{-k}, W_1] = (1 - q^{-2})(\tilde{G}_{k+1} - G_{k+1}),
\]
\[
[W_0, G_{k+1}]_q = [\tilde{G}_{k+1}, W_0]_q = (q - q^{-1})W_{-k-1},
\]
\[
[G_{k+1}, W_1]_q = [W_1, \tilde{G}_{k+1}]_q = (q - q^{-1})W_{k+2}.
\]
Lemma

For $k, \ell \in \mathbb{N}$ the following relations hold in the $q$-shuffle algebra $\mathcal{V}$:

\[
\begin{align*}
[W_{-k}, W_{-\ell}] & = 0, \quad [W_{k+1}, W_{\ell+1}] = 0, \\
[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] & = 0, \\
[W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] & = 0, \\
[W_{-k}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{-\ell}] & = 0, \\
[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] & = 0, \\
[W_{k+1}, \tilde{G}_{\ell+1}] + [\tilde{G}_{k+1}, W_{\ell+1}] & = 0, \\
[G_{k+1}, G_{\ell+1}] & = 0, \quad [\tilde{G}_{k+1}, \tilde{G}_{\ell+1}] = 0, \\
[\tilde{G}_{k+1}, G_{\ell+1}] + [G_{k+1}, \tilde{G}_{\ell+1}] & = 0.
\end{align*}
\]
Lemma

For $k, \ell \in \mathbb{N}$ the following relations hold in the $q$-shuffle algebra $\mathbb{V}$:

\begin{align*}
[W_{-k}, G_\ell]_q &= [W_{-\ell}, G_k]_q, \\
[\tilde{G}_k, W_-]_q &= [\tilde{G}_\ell, W_-]_q, \\
[G_k, \tilde{G}_{\ell+1}] - [G_\ell, \tilde{G}_{k+1}] &= q[W_{-\ell}, W_{k+1}]_q - q[W_{-k}, W_{\ell+1}]_q, \\
[\tilde{G}_k, G_{\ell+1}] - [\tilde{G}_\ell, G_{k+1}] &= q[W_{\ell+1}, W_-]_q - q[W_{k+1}, W_-]_q, \\
[G_{k+1}, \tilde{G}_{\ell+1}]_q - [G_{\ell+1}, \tilde{G}_{k+1}]_q &= q[W_{-\ell}, W_{k+2}] - q[W_{-k}, W_{\ell+2}], \\
[\tilde{G}_{k+1}, G_{\ell+1}]_q - [\tilde{G}_{\ell+1}, G_{k+1}]_q &= q[W_{\ell+1}, W_{-k-1}] - q[W_{k+1}, W_{-\ell-1}].
\end{align*}
Lemma

For \( n \geq 1 \),

\[
\sum_{k=0}^{n} G_k \ast \tilde{G}_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{-k} \ast W_{n-k} q^{n-1-2k},
\]

\[
\sum_{k=0}^{n} G_k \ast \tilde{G}_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{n-k} \ast W_{-k} q^{n-1-2k},
\]

\[
\sum_{k=0}^{n} \tilde{G}_k \ast G_{n-k} q^{n-2k} = q \sum_{k=0}^{n-1} W_{n-k} \ast W_{-k} q^{2k+1-n},
\]

\[
\sum_{k=0}^{n} \tilde{G}_k \ast G_{n-k} q^{2k-n} = q \sum_{k=0}^{n-1} W_{-k} \ast W_{n-k} q^{2k+1-n}.
\]
Obtaining the alternating words from $x, y$

**Lemma**

*Using the equations below, the alternating words in $\nabla$ are recursively obtained from $x, y$ in the following order:*

$$W_0, \ W_1, \ G_1, \ \tilde{G}_1, \ W_{-1}, \ W_2, \ G_2, \ \tilde{G}_2, \ ...$$

*We have $W_0 = x$ and $W_1 = y$. For $n \geq 1$,*

$$G_n = \frac{q \sum_{k=0}^{n-1} W_{-k} \ast W_{n-k} q^{n-1-2k} - \sum_{k=1}^{n-1} G_k \ast \tilde{G}_{n-k} q^{n-2k}}{q^n + q^{-n}} + \frac{W_n \ast W_0 - W_0 \ast W_n}{(1 + q^{-2n})(1 - q^{-2})},$$

$$\tilde{G}_n = G_n + \frac{W_0 \ast W_n - W_n \ast W_0}{1 - q^{-2}},$$

$$W_{-n} = \frac{q W_0 \ast G_n - q^{-1} G_n \ast W_0}{q - q^{-1}},$$

$$W_{n+1} = \frac{q G_n \ast W_1 - q^{-1} W_1 \ast G_n}{q - q^{-1}}.$$
Corollary

$U$ contains the alternating words.
We will use the alternating words to obtain a PBW basis for $U$.

We wont use all the alternating words, because some of them can be written in terms of the others.

Over the next five slides, we show how to write $\{G_n\}_{n=1}^{\infty}$ in terms of $\{W_{-n}\}_{n=0}^{\infty}$, $\{W_{n+1}\}_{n=0}^{\infty}$, $\{\tilde{G}_n\}_{n=1}^{\infty}$. 
At this point, it is convenient to make a change of variables.

**Definition**

Define elements \( \{D_n\}_{n=0}^{\infty} \) in \( \mathcal{V} \) such that \( D_0 = 1 \) and for \( n \geq 1 \),

\[
D_0 \ast \tilde{G}_n + D_1 \ast \tilde{G}_{n-1} + \cdots + D_n \ast \tilde{G}_0 = 0.
\]
A change of variables, cont.

**Example**

We have

\[ D_1 = -\tilde{G}_1, \]
\[ D_2 = \tilde{G}_1 \star \tilde{G}_1 - \tilde{G}_2, \]
\[ D_3 = 2\tilde{G}_1 \star \tilde{G}_2 - \tilde{G}_1 \star \tilde{G}_1 \star \tilde{G}_1 - \tilde{G}_3, \]
\[ D_4 = \tilde{G}_1 \star \tilde{G}_1 \star \tilde{G}_1 \star \tilde{G}_1 + 2\tilde{G}_1 \star \tilde{G}_3 + \tilde{G}_2 \star \tilde{G}_2 - 3\tilde{G}_1 \star \tilde{G}_1 \star \tilde{G}_2 \star \tilde{G}_1 - \tilde{G}_4 \]

and

\[ \tilde{G}_1 = -D_1, \]
\[ \tilde{G}_2 = D_1 \star D_1 - D_2, \]
\[ \tilde{G}_3 = 2D_1 \star D_2 - D_1 \star D_1 \star D_1 - D_3, \]
\[ \tilde{G}_4 = D_1 \star D_1 \star D_1 \star D_1 + 2D_1 \star D_3 + D_2 \star D_2 - 3D_1 \star D_1 \star D_2 - D_4. \]
A change of variables, cont.

The following two results clarify how the $D_n$ are related to the $\tilde{G}_n$.

**Lemma**

For $n \geq 1$ the following hold in the q-shuffle algebra $\mathbb{V}$.

(i) $D_n$ is a homogeneous polynomial in $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_n$ that has total degree $n$, where we view each $\tilde{G}_i$ as having degree $i$.

(ii) $\tilde{G}_n$ is a homogeneous polynomial in $D_1, D_2, \ldots, D_n$ that has total degree $n$, where we view each $D_i$ as having degree $i$. 

The alternating PBW basis for the positive part of $U_q(\widehat{sl}_2)$ 

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A change of variables, cont.

**Lemma**

The following coincide:

(i) the subalgebra of the q-shuffle algebra $\bigvee$ generated by $\{D_n\}_{n=1}^{\infty}$;

(ii) the subalgebra of the q-shuffle algebra $\bigvee$ generated by $\{\tilde{G}_n\}_{n=1}^{\infty}$. 

The alternating PBW basis for the positive part of $U_q(\hat{sl}_2)$
Eliminating the $G_n$

Using our earlier relations I–IV we obtain the following result.

**Lemma**

For $n \in \mathbb{N}$ we have

$$G_n = q^{2n} D_n + q^2 \sum_{i+j+k+1=n \atop i,j,k \geq 0} W_{-i} \ast D_j \ast W_{k+1}.$$ 

Because of this result, we eliminate the $G_n$ from consideration, as we construct the alternating PBW basis.
The alternating PBW basis for $U$

**Theorem**

A PBW basis for $U$ is obtained by the elements

$$\{W_{-i}\}_{i \in \mathbb{N}}, \quad \{\tilde{G}_{j+1}\}_{j \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}$$

in any linear order $<$ that satisfies

$$W_{-i} < \tilde{G}_{j+1} < W_{k+1} \quad i, j, k \in \mathbb{N}.$$ 

**Definition**

The above PBW basis is called **alternating**.
Comparing the Damiani PBW basis and the alternating PBW basis

Next we explain how the Damiani PBW basis is related to the alternating PBW basis.

We adopt the following point of view. Instead of working directly with the Damiani PBW basis elements, we will work with the closely related elements \( \{xC_n\}_{n=0}^{\infty}, \{C_ny\}_{n=0}^{\infty}, \{C_n\}_{n=1}^{\infty} \).
Comparing $\{C_n\}_{n=1}^{\infty}$, $\{\tilde{G}_n\}_{n=1}^{\infty}$, $\{D_n\}_{n=1}^{\infty}$

Earlier we saw how the elements $\{\tilde{G}_n\}_{n=1}^{\infty}$ and $\{D_n\}_{n=1}^{\infty}$ are related.

We now explain how these elements are related to $\{C_n\}_{n=1}^{\infty}$.

**Theorem**

For $n \geq 1$,

$$C_n = (-1)^n \sum_{i=0}^{n} q^{2i-n} D_i \star D_{n-i}$$
Comparing \( \{ C_n \}_{n=1}^{\infty}, \{ \tilde{G}_n \}_{n=1}^{\infty}, \{ D_n \}_{n=1}^{\infty} \), cont.

**Corollary**

For \( n \geq 1 \) the following hold in the q-shuffle algebra \( \mathbb{V} \).

(i) \( C_n \) is a homogeneous polynomial in \( D_1, D_2, \ldots, D_n \) that has total degree \( n \), where we view each \( D_i \) as having degree \( i \).

(ii) \( D_n \) is a homogeneous polynomial in \( C_1, C_2, \ldots, C_n \) that has total degree \( n \), where we view each \( C_i \) as having degree \( i \).
Comparing $\{C_n\}_{n=1}^{\infty}$, $\{\tilde{G}_n\}_{n=1}^{\infty}$, $\{D_n\}_{n=1}^{\infty}$, cont.

Corollary

For $n \geq 1$ the following hold in the $q$-shuffle algebra $\mathcal{V}$.

(i) $C_n$ is a homogeneous polynomial in $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_n$ that has total degree $n$, where we view each $\tilde{G}_i$ as having degree $i$.

(ii) $\tilde{G}_n$ is a homogeneous polynomial in $C_1, C_2, \ldots, C_n$ that has total degree $n$, where we view each $C_i$ as having degree $i$. 

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Corollary

The following (i)–(iii) coincide:

(i) the subalgebra of the q-shuffle algebra $\nabla$ generated by $\{C_n\}_{n=1}^{\infty}$;

(ii) the subalgebra of the q-shuffle algebra $\nabla$ generated by $\{D_n\}_{n=1}^{\infty}$;

(iii) the subalgebra of the q-shuffle algebra $\nabla$ generated by $\{\tilde{G}_n\}_{n=1}^{\infty}$.
Comparing the Damiani PBW basis and the alternating PBW basis

We now write the \( \{xC_n\}_{n=0}^{\infty} \) and \( \{C_ny\}_{n=0}^{\infty} \) in the alternating PBW basis.

**Theorem**

For \( n \in \mathbb{N} \),

\[
xC_n = (-1)^n q^{-n} \sum_{i=0}^{n} W_{-i} \ast D_{n-i},
\]

\[
C_ny = (-1)^n q^{-n} \sum_{i=0}^{n} D_{n-i} \ast W_{i+1}.
\]
Comparing the Damiani PBW basis and the alternating PBW basis, cont.

We now write the \(\{W_{-n}\}_{n=0}^{\infty}\) and \(\{W_{n+1}\}_{n=0}^{\infty}\) in the Damiani PBW basis.

**Theorem**

For \(n \in \mathbb{N}\),

\[
W_{-n} = \sum_{i=0}^{n} (-1)^i q^i (xC_i) \ast \tilde{G}_{n-i},
\]

\[
W_{n+1} = \sum_{i=0}^{n} (-1)^i q^i \tilde{G}_{n-i} \ast (C_iy).
\]
First we recalled the Damiani PBW basis for $U_q^+$. 

Next we expressed the Damiani PBW basis elements in closed form, using a $q$-shuffle algebra. 

Using the $q$-shuffle algebra, we obtained an attractive new PBW basis for $U_q^+$, said to be alternating. 

Finally we described how the Damiani PBW basis is related to the alternating PBW basis.

THANK YOU FOR YOUR ATTENTION!