4.2 The vector space $\mathbb{R}^n$ and subspaces

Earlier we discussed the vector space $\mathbb{R}^3$.

We now generalize

$$\mathbb{R}^3 \rightarrow \mathbb{R}^n$$

Def: the vector space $\mathbb{R}^n$ consists of the set of column $n$-vectors $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ together with the operations of vector addition and scalar mult.

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Vector addition and scalar mult obey these axioms (routinely checked):
\[ u + v = v + u \]
\[ u + (v + w) = (u + v) + w \]
\[ u + 0 = 0 + u = u \]
\[ u + (-u) = (-u) + u = 0 \]
\[ a(u + v) = au + av \]
\[ (a + b)u = au + bu \]
\[ a(bu) = (ab)u \]
\[ (1)u = u \]

We use the above axioms to define an abstract vector space...
Def A vector space (over IR) is a set V (of vectors) together with a binary operation + and a scalar mult, that obey the above axioms.

Ex Given pos integers m, n def V = set of all m x n matrices with entries in IR, then V together with the usual matrix addition and scalar mult is vector space.

Ex Let V = set of all functions f:IR->IR that are continuous everywhere. Define +:
Given f, g ∈ V the function f+g sends x → f(x) + g(x) f+g ∈ V since the sum of two contin functions is contin.

Define sc mult:
Given f ∈ V Given scalar a af ∈ V since scalar mult of contin function is contin.
Ex. Given a vector space \( V \)

Let \( W \) be a nonempty subset of \( V \) such that:

- Given any vectors \( u, v \) in \( W \), then \( u + v \) is in \( W \),
  
  "closure under addition"

- Given any vector \( u \) in \( W \), and given any scalar \( c \),
  then \( cu \) is in \( W \).

"closure under scalar mult"

Then the set \( W \), together with the addition operation and scalar mult inherited from \( V \), form a vector space, called a \underline{subspace of} \( V \).
Examples of subspaces

Ex  Given a vector space V.

Define

\[ W = \text{subset of } V \text{ consisting of single element } 0 \]

Then \( W \) is a subspace of \( V \)

"The zero subspace"

pf \ Check \( W \) is closed under +:

\[ 0 + 0 = 0 \]

Check \( W \) is closed under scalar mult:

\[ c(0) = 0 \text{ for all } c \in \mathbb{R} \]

Ex  Given a vector space \( V \)

Define

\[ W = V \]

Then \( W \) is a subspace of \( V \)
Ex. Given a vector space \( V \)

Given a vector \( u \in V \)

Define
\[
W = \left\{ cu \mid c \in \mathbb{R} \right\}
\]

"set of all scalar multiples of \( u \)

Then \( W \) is a subspace of \( V \)

pf. check \( W \) is closed under +:

Given two vectors in \( W \), say,

\[ cu, \quad cv \]

Then
\[
cu + cv = (c + c')u
\]

\[ = sc \text{ multiple of } u
\]

\[ \in W \]

Check \( W \) is closed under scalar multi:

Given vector in \( W \), say,

\[ cu \]

Given scalar \( a \)

\[
a(cu) = (ac)u
\]

\[ = \text{ scalar multiple of } u
\]

\[ \in W \]
Ex. Given a vector space $V$

Given two vectors $u, v \in V$

Define

$$W = \left\{ au + bv \mid a, b \in \mathbb{R} \right\}$$

"set of all linear combinations of $u$ and $v$"

"span of $u$ and $v$"

Then $W$ is a subspace of $V$.

pf. Check $W$ is closed under $+$:

Given two vectors in $W$, say $au + bv$, $a'u + bv'$

Then

$$(au + bv) + (a'u + bv') = (a + a')u + (b + b')v$$

= linear comb of $u, v$

$\in W$  

Check $W$ is closed under scalar mult:

Given vector in $W$, say $au + bv$. Given scalar $c$

$$c(au + bv) = (ac)u + (bc)v$$

= linear comb of $u, v$

$\in W$
Ex. Given \( m \times n \) matrix \( A \).

Define

\[
W = \text{set of all solutions to } A\mathbf{x} = \mathbf{0}
\]

Then \( W \) is a subspace of the vector space \( \mathbb{R}^n \).

Proof (pf)

Check \( W \) is closed under \(+\).

Given two vectors in \( W \), say \( u, v \).

So \( A\mathbf{u} = \mathbf{0}, \quad A\mathbf{v} = \mathbf{0} \).

Then

\[
A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}
\]

So \( \mathbf{u} + \mathbf{v} \in W \).

Check \( W \) is closed under scalar multiples.

Given \( \mathbf{u} \in W \) so \( A\mathbf{u} = \mathbf{0} \).

Given scalar \( c \).

\[
A(c\mathbf{u}) = c(A\mathbf{u}) = c\mathbf{0} = \mathbf{0}
\]

So \( c\mathbf{u} \in W \). \( \square \)
Cautions. Given an $m \times n$ matrix $A$

Given a **nonzero** $b \in \mathbb{R}^m$

Define

$$W = \text{set of all solutions to } A\mathbf{x} = b$$

Then $W$ is **NOT** a subspace of $\mathbb{R}^n$. Indeed, $W$ is not closed under +:

Given $u, v \in W, \text{ so }$

$$A\mathbf{u} = b \quad A\mathbf{v} = b$$

Then

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = b + b = 2b \neq b$$

So $\mathbf{u} + \mathbf{v} \notin W$

Also, $W$ is not closed under scalar multiplication.
Describe the nullspace of

$$A = \begin{bmatrix}
    1 & -4 & 1 & -4 \\
    1 & 2 & 1 & 8 \\
    1 & 1 & 1 & 6
\end{bmatrix}$$

Sol.

Find all the sols to

$$A \vec{x} = \vec{0}$$

$$\vec{x} = \begin{bmatrix}
    x \\
    y \\
    z \\
    w
\end{bmatrix}$$

App1. GJ

$$\begin{bmatrix}
    1 & -4 & 1 & -4 \\
    1 & 2 & 1 & 8 \\
    1 & 1 & 1 & 6
\end{bmatrix}$$

$$\begin{bmatrix}
    1 & -4 & 1 & -4 \\
    0 & 6 & 0 & 12 \\
    0 & 5 & 0 & 10
\end{bmatrix}$$

$$r_2 = r_2 - r_1$$

$$r_3 = r_3 - r_1$$

$$\begin{bmatrix}
    1 & -4 & 1 & -4 \\
    0 & 1 & 0 & 2 \\
    0 & 5 & 0 & 10
\end{bmatrix}$$

$$r_2 = \frac{1}{6} r_2$$
\[
\begin{bmatrix}
1 & -4 & 1 & -4 \\
0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[ r_3 = r_3 - 5r_2 \]

**Backsolve**

**Leading vars:** \( x, y \)

\[ z = x \]
\[ w = t \]

\[ y = -2t \]

\[ x = 4y - z + 4w \]
\[ = -4t \]

\[
\begin{bmatrix}
 x \\
y \\
z \\
w
\end{bmatrix}
= \alpha
\begin{bmatrix}
-1 \\
0 \\
1 \\
0
\end{bmatrix}
+ t
\begin{bmatrix}
-4 \\
-2 \\
0 \\
1
\end{bmatrix}
\]

**Null space of \( A \)**

\[ \{ su + tv \mid s, t \in \mathbb{R} \} \]

\[ = \text{set of all linear combinations of } u, v \]
Ex

Given a vector space $V$

Given two subspaces $U, W$

Show that the intersection

$U \cap W$

is a subspace of $V$

Sol. • Check $U \cap W$ is closed under +

Given two vectors in $U \cap W$, denoted $u, v$

So $u \in U, v \in U, u \in W, v \in W$

show $u + v \in U \cap W$

$u + v \in U$

(by closure under $+$)

$u + v \in W$

(by closure under $+$)

So $u + v \in U \cap W$

• Check $U \cap W$ is closed under scalar mult

Given vector $u \in U \cap W$, scalar $c$

show $cu \in U \cap W$

$cu \in U$

(by closure under scalar mult)

$cu \in W$

(by closure under scalar mult)

So $cu \in U \cap W$
Describe the null space of 

\[ A = \begin{bmatrix}
1 & 5 & 1 & -8 \\
2 & 5 & 0 & -5 \\
2 & 4 & 1 & -9
\end{bmatrix} \]

Find all the null to 

\[ A \mathbf{x} = \mathbf{0} \]

\[ \mathbf{x} = \begin{bmatrix}
x \\
y \\
z
\end{bmatrix} \]

Apply GJ

\[ \begin{bmatrix}
1 & 5 & 1 & -8 \\
2 & 5 & 0 & -5 \\
2 & 4 & 1 & -9
\end{bmatrix} \]

\[ r_2' = r_2 - 2r_1 \]

\[ r_3' = r_3 - 2r_1 \]

\[ \begin{bmatrix}
1 & 5 & 1 & -8 \\
0 & -5 & -2 & 11 \\
0 & -3 & -1 & 7
\end{bmatrix} \]

\[ r_2' = r_2 - 2r_1 \]

\[ \begin{bmatrix}
1 & 5 & 1 & -8 \\
0 & 1 & 0 & -3 \\
0 & -3 & -1 & 7
\end{bmatrix} \]

\[ r_3' = r_3 + 3r_2 \]

\[ \begin{bmatrix}
1 & 5 & 1 & -8 \\
0 & 1 & 0 & -3 \\
0 & 0 & -1 & -2
\end{bmatrix} \]

\[ r_3' = -r_3 \]
Backsolve

$x, y, z$ leading vars

$$w = t \quad \text{free}$$

$$t = -2t$$

$$y = 3t$$

$$x = -5y - z + 8w$$

$$= -5t$$

$$\begin{bmatrix}
  x \\
  y \\
  z \\
  w
\end{bmatrix} = t
\begin{bmatrix}
  -5 \\
  3 \\
  -2 \\
  1
\end{bmatrix}$$

Null space $\mathbf{N}(A) = \left\{ b\mathbf{u} \mid b \in \mathbb{R} \right\}$

$$= \text{set of scalar multiples of } \mathbf{u}$$