4.4 Bases and Dimension for vector spaces

Given a vector space $V$

Given vectors in $V$: $v_1, v_2, \ldots, v_n$

These vectors form a basis for $V$ provided both

(i) $v_1, \ldots, v_n$ are linearly independent

(ii) $\text{Span}(v_1, \ldots, v_n) = V$

Example

Recall the standard unit vectors in $\mathbb{R}^n$

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$$

Then $e_1, e_2, \ldots, e_n$ form a basis for $\mathbb{R}^n$

"Standard basis for $\mathbb{R}^n"
the standard basis is not the only basis for $\mathbb{R}^n$. To see this, we extend a VM from Section 4.3.

**Thm.** Given $n$ vectors in $\mathbb{R}^n$, say $v_1, v_2, \ldots, v_n$.

Define an $n \times n$ matrix $A$ such that for $1 \leq i \leq n$,

$$\text{col}_i(A) = v_i$$

Then, the following are equivalent:

1. $v_1, v_2, \ldots, v_n$ are linearly independent.
2. $A$ is invertible.
3. $\text{span}(v_1, \ldots, v_n) = \mathbb{R}^n$.
4. $v_1, v_2, \ldots, v_n$ is a basis for $\mathbb{R}^n$.

**pf.**

(i) ⇔ (iii) ⇔ (i): Shown in Sec 4.3.

(i), (iii) → (iv): This is def of basis.

(iv) → (i): By def of basis.
**Ex** For vector space $\mathbb{R}^3$ define

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, $v_3 = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

Determine if $\{v_1, v_2, v_3\}$ form a basis for $\mathbb{R}^3$.

**Sol** Define matrix $A$:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{vmatrix} = -1 \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} + 0$$

$$= -2 + 4$$

$$= 2$$

$A$ is invertible, so $\{v_1, v_2, v_3\}$ form a basis for $\mathbb{R}^3$. 
Ex  

Let $V$ = subspace of $\mathbb{R}^3$ consisting of all vectors

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}
\]

$x - 2y + 5z = 0$

Find a basis for $V$.

Sol  

Given $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in V$.

$x - 2y + 5z = 0$

View $y$, $z$ as free

\[
y = 4, \quad z = 6, \quad x \text{ at free}
\]

\[
x = 24 - 5t
\]

\[
\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix} = t \begin{bmatrix}
  -5 \\
  0 \\
  1
\end{bmatrix}
\]

\[
\text{Span } \begin{bmatrix} u, v \end{bmatrix} = V
\]

$u, v$ linearly, since one is not a scalar multiple of the other

So $u, v$ is a basis for $V$.
Next goal: Given vector space $V$

show any two bases for $V$ have the
same number of vectors.

**Thm:** Given vector space $V$

Given a basis for $V$, say

$v_1, v_2, \ldots, v_n$

then any set of more than $n$ vectors in $V$ is linearly depen

pf

Given vectors in $V$

$w_1, w_2, \ldots, w_m$ in $V$

Show these are linear depen

Find scalars $c_1, c_2, \ldots, c_m$ (not all 0) such that

$c_1 w_1 + c_2 w_2 + \cdots + c_m w_m = 0$

Write each $w_i$ in terms of $v_1, \ldots, v_n$.
\[ w_1 = a_{11} v_1 + a_{21} v_2 + \cdots + a_{n1} v_n \]
\[ w_2 = a_{12} v_1 + a_{22} v_2 + \cdots + a_{n2} v_n \]
\[ \vdots \]
\[ w_m = a_{1m} v_1 + a_{2m} v_2 + \cdots + a_{nm} v_n \]

(\text{X}) \quad \text{becomes}

\[
\begin{pmatrix}
  a_{11} c_1 + a_{12} c_2 + \cdots \\
  a_{21} c_1 + a_{22} c_2 + \cdots \\
  \vdots \\
  a_{n1} c_1 + a_{n2} c_2 + \cdots \\
\end{pmatrix}
\begin{pmatrix}
  v_1 \\
  v_2 \\
  \vdots \\
  v_n \\
\end{pmatrix}
\]

\[ = 0 \]

But \( v_{n+1} \) is linearly independent so each coeff is 0:

\[
\begin{align*}
  a_{11} c_1 + a_{12} c_2 + \cdots + a_{1m} c_m &= 0 \\
  a_{21} c_1 + a_{22} c_2 + \cdots + a_{2m} c_m &= 0 \\
  \vdots \quad \vdots \quad \vdots \\
  a_{n1} c_1 + a_{n2} c_2 + \cdots + a_{nm} c_m &= 0 
\end{align*}
\]

Linear system (\text{X}) has more variables than equations, so there exists a non-trivial sol \( c_1, c_2, \ldots \)
**Thm 2**  Given a vector space $V$, then any two bases for $V$ have the same number of vectors.

**pf**

Call the bases $v_1, v_2, ..., v_n$ and $w_1, w_2, ..., w_m$.

Show $n = m$.

Interchanging the bases $w_i$ across $v_i$, we find $n = m$.

Suppose $n < m$. Then $w_1, ..., w_m$ are linearly independent by prev Thm.

So $n = m$.

$\square$
Def: Given a vector space $V$

By the definition of $V$, we mean the number of vectors in any basis for $V$

$\text{Ex}$ the dimension of $\mathbb{R}^n$ is $n$

$\text{Caut} \text{ion} \quad \text{For some vector spaces the dimension} = \infty$

$\text{Ex}$ let $V$ denote the vector space of all polynomials in the variable $x$

$\left[ \begin{array}{c}
1 + x - x^2 \\
2x - x^3 \\
1 + x^10 + x^{20} \\
\end{array} \right]$

One checks that

$\{1, x, x^2 \}$ ---

is a basis for $V$.

$(* \text{I}$ has $\infty$ vectors so $\dim V = \infty$)
Ex

Given linear system

\[ X_1 - 3X_2 + 2X_3 - 4X_4 = 0 \]
\[ 2X_1 - 5X_2 + 7X_3 - 3X_4 = 0 \]

Find a basis for the solution space.

Sol

Solve the system

\[
\begin{bmatrix}
1 & -3 & 2 & -4 \\
2 & -5 & 7 & -3
\end{bmatrix}
\]

coeft matrix

Apply GJ

\[
\begin{bmatrix}
1 & -3 & 2 & -4 \\
0 & 1 & 3 & 5
\end{bmatrix}
\]

\[ r_2 = r_2 - 3r_1 \]

\[
\begin{bmatrix}
1 & 0 & 11 & 11 \\
0 & 1 & 3 & 5
\end{bmatrix}
\]

\[ r_1 = r_1 + 3r_2 \]

Back solve:

\[ x_3 = 4 \]
\[ x_4 = 6 \]
\[ x_1 \text{ and free} \]
\[ x_2 = -3 - 5x_4 \]
\[ x_1 = -11x_4 - 11x_4 \]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= A
\begin{bmatrix}
-11 \\
-3 \\
1 \\
0
\end{bmatrix}
+ t
\begin{bmatrix}
-11 \\
-5 \\
0 \\
1
\end{bmatrix}
\]

\[ u, v \text{ span sol space} \]
\[ u, v \text{ linear indep} \]
\[ u, v \text{ is basis for sol space} \]
Theorem 3. Given a finite dimensional vector space $V$.

Given a spanning set for $V$:

$$v_1, v_2, \ldots, v_n$$

Then there exists a subset of $(* \setminus v_i)$ that forms a basis for $V$.

Proof. Suppose $(*)$ is linearly independent. Then $(*)$ is a basis for $V$.

Suppose $(*)$ is linearly dependent.

So there exist scalars $c_1, c_2, \ldots, c_n$ (not all 0) such that

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$$

There exists $i$ (1 ≤ $i$ ≤ n) such that $c_i \neq 0$.

Now $v_i \in \text{Span}(v_1, v_2, v_3, \ldots, v_n)$

Now remove $v_i$ from list $(*).$ Modified list still spans $V$.

Iterate - procedure yields a basis for $V$. \hfill \square
Ex for $\mathbb{R}^3$ define

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

let

$$V = \text{Span}(v_1, v_2, v_3, v_4)$$

Find a basis for $V$

Sol

Find the linear dependencies among $v_1, v_2, v_3, v_4$:

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \end{bmatrix}$$

$$GJ$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \end{bmatrix}$$

$r_1' = r_1 - r_4$

$r_2' = r_2 - r_3$

$r_3' = r_3 - r_4$

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r_4' = r_4 - 2r_3$
Backsolve

\[ c_5 = 1 \quad c_4 = t \quad x = t \text{ free} \]

\[ c_2 = 0 \quad c_1 = -2t \]

\[
\begin{pmatrix}
  c_5 \\
  c_4 \\
  c_3 \\
  c_1
\end{pmatrix} = t \begin{pmatrix}
  2 \\
  1 \\
  1 \\
  0
\end{pmatrix} + \begin{pmatrix}
  -2 \\
  -1 \\
  0 \\
  1
\end{pmatrix}
\]

Each vector gives a dependency among \( v_1, v_2, v_3, v_4 \)

\[-v_1 - v_2 + v_3 = 0 \quad \text{"discard } v_3 \text{"} \]

\[-2v_1 - v_2 + v_4 = 0 \quad \text{"discard } v_4 \text{"} \]

\[ \text{Span } (v_1, v_2) = \mathbb{V} \]

\[ v_1, v_2 \text{ line independent} \]

\[ v_1, v_2 \text{ basis for } \mathbb{V} \]
Given vector space $V$ with finite dimension $n$.

Given $n$ vectors in $V$: $v_1, v_2, \ldots, v_n$.

Then the following are equivalent:

(i) $v_1, v_2, \ldots, v_n$ are linearly independent.

(ii) $\text{Span}(v_1, \ldots, v_n) = V$.

(iii) $v_1, v_2, \ldots, v_n$ is a basis for $V$.

Proof:

(i) $\Rightarrow$ (iii): Suppose $\text{Span}(v_1, \ldots, v_n) \neq V$.

Then there exists a vector in $V$ that is not in $\text{Span}(v_1, \ldots, v_n)$.

Call this vector $v$.

Then $v_1, v_2, \ldots, v_n, v$ are linearly independent.

This contradicts that $v_1, v_2, \ldots, v_n$ are linearly independent.

(iii) $\Rightarrow$ (ii): By Thm 3, there exists a subset of $v_1, v_2, \ldots, v_n$ that is a basis for $V$.

By Thm 2, this subset consists of all of $v_1, v_2, \ldots, v_n$.

(i), (ii) $\Rightarrow$ (iii): Def of basis.

(iii) $\Rightarrow$ (i): Def of basis.
Theorem. Given a finite dimensional vector space $V$,
Given $m$ linearly independent vectors in $V$:

$v_1, v_2, \ldots, v_n$

Then there exists a basis for $V$ that contains $(*).$

Proof.

Suppose $(*).$ spans $V$, then $(*).$ is a basis for $V$; done.

Suppose $(*).$ does not span $V$, then there exists a vector in $V$ that is not in

$\text{Span}(v_1, v_2, \ldots, v_n)$

Call this vector $v_m$.

So $v_1, v_2, \ldots, v_m$

are linearly independent.

Add $v_m$ to $(*).

Repeat (process ends by Rank $\geq 1.$) to get

a basis for $V$ that contains $(*).$