We now consider differential equations of the form

\[
\frac{dy}{dx} = f(x, y)
\]

There is no uniform method that gives the solution in closed form.

But we can use the slope field to estimate the solutions.

Example: Describe the solutions to

\[
\frac{dy}{dx} = x^2 + y^2 - 1
\]

Solution: Sketch slope field.
- at origin \((x, y) = (0, 0)\) slope is \(-1\)

- at all pts \((x, y)\) with \(x^2 + y^2 = 1\), slope is 0

Circle abt origin with radius 1

- For any radius \(r\), consider circle abt origin with radius \(r\)

For each pt \((x, y)\) on this circle

\[x^2 + y^2 = r^2\]

So slope at \((x, y)\) is \(r^2 - 1\)

<table>
<thead>
<tr>
<th>(r)</th>
<th>(r^2 - 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>(\frac{1}{2})</td>
<td>-(\frac{3}{4})</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
</tbody>
</table>
Each curve gives a sol to *.

Ex. For the initial value problem

\[ \frac{dy}{dx} = x^2 + y^2 - 1, \quad y(0) = 0 \]

Estimate \( y(2) \)

Sol. The curve thru origin goes close to (2,1), so \( y(2) \approx 1 \).
Ex. Describe the sols to

\[
\frac{dy}{dx} = -y + 2
\]

Sol sketch slope field

- On line \( y = 0 \) slope is 2
- On line \( y = 1 \) slope is 1
- On line \( y = 2 \) slope is 0
- On line \( y = 3 \) slope is -1

Each curve is a sol.
the function

\[ y = 2 \]

is a particular solution.

For any solution curve, as \( x \) grows, \( y \) approaches 2.

Ex. For the init value problem

\[ \frac{dy}{dx} = -y + 2 \quad y(0) = 10 \]

Estimate \( y(1000) \)

SOL \( y(1000) \approx 2 \)
For the previous two examples,

For any pt \((a,b)\) in \(\mathbb{R}^2\), there is a unique solution curve thru that point.

"Start at \((a,b)\) and follow the arrows in the slope field."

Caution For a given diff equation

\[
\frac{dy}{dx} = f(x,y)
\]

and a given pt \((a,b)\) in \(\mathbb{R}^2\)

- A solution curve thru \((a,b)\) may not exist.
- The solution curve thru \((a,b)\) might not be unique.

The following examples illustrate what can go wrong.
Ex. Find all sols to

\[ \frac{dy}{dx} = \frac{y}{x} \]

Sol. First sketch slope field.

- On line \( y = 0 \) slope is 0
- On line \( y = x \) slope is 1
- On line \( y = -x \) slope is -1
- On line \( y = 2x \) slope is 2
- For any \( A \in \mathbb{R} \), on the line \( y = Ax \) the slope is \( A \)
It appears that each (non-vertical) line through origin gives a solution curve

General solution should be

\[ y = Cx \quad \text{with} \quad C = \text{constant} \]

Check

\[ \frac{dy}{dx} = C \]

\[ = \frac{y}{x} \quad \checkmark \]

Observe

- For any point \((a, b)\) with \(a = 0\) and \(b \neq 0\)
  - There is no solution curve through \((a, b)\)

- At the origin \((0, 0)\) there are \(\infty\) many solution curves through \((0, 0)\)

- For \((a, b)\) with \(b \neq 0\) there is unique solution curve through \((a, b)\)

Essential problem:

\[ \frac{dy}{dx} = \frac{y}{x} \quad \text{RHS is not defined for} \quad x = 0 \]
Ex. Describe the solutions to

\[ \frac{dy}{dx} = -\sqrt{1 - y^2} \]

Sol. Abs: At any pt. \((x, y)\) in \(\mathbb{R}^2\), no solution curve thru \((x, y)\) unless \(-1 \leq y \leq 1\)

Sketch slope field:

- On the line \(y = 0\) slope is \(-1\)
  - \(y = 0\) ........ \(-1\)

--Trig views:
  - \(\sqrt{1 - y^2} = \sin \theta\)
  - \(y = \cos \theta\)
  - \(-1 \leq y \leq 1\) view \(b = \cos \theta\) \((0 \leq \theta < \pi)\)

On line \(y = b\) slope is \(-\sin \theta\)
Slope field:

\[ y' = f(x, y) \]

Solutions:

\[ y = g(x) \]
Precise description of sets:

Pick any \( c \in \mathbb{R} \)

Define a function \( y \) by:

\[
y = \begin{cases} 
\cos(x - c) & \text{if } c \leq x \leq c + \pi \\
1 & \text{if } x < c \\
-1 & \text{if } x > c + \pi
\end{cases}
\]
Check that the function $y$ is a particular sol.

Find $\frac{dy}{dx}$

For $c \leq x \leq c + \pi$,

$$y = \cos(x - c)$$

$$\frac{dy}{dx} = -\sin(x - c)$$

$$= -\sqrt{1 - \cos^2(x - c)}$$

$$= -\sqrt{1 - y^2}$$

For $x < c$,

$$y = 1$$

$$\frac{dy}{dx} = 0$$

$$= -\sqrt{1 - 1^2}$$

$$= -\sqrt{1 - y^2}$$

For $x > c + \pi$,

$$y = -1$$

$$\frac{dy}{dx} = 0$$

$$= -\sqrt{1 - (-1)^2}$$

$$= -\sqrt{1 - y^2}$$
We have given some particular sols
the lines \( y = 1 \), \( y = -1 \) are also particular sols.

Turns out there are no other sols.

So for any pt \((a,b)\) in \(\mathbb{R}^2\)

1. If \( b > 1 \) or \( b < -1 \) then NO sol passes thru \((a,b)\)

2. If \( b = 1 \) or \( b = -1 \) then \( \infty \) sols --

3. If \( -1 < b < 1 \) then unique sol --
Then given a point \((a, b)\) in \(\mathbb{R}^2\) and consider the initial value problem

\[
\frac{dy}{dx} = f(x, y) \quad y(a) = b
\]

Assume both

\[
f(x, y), \quad \frac{\partial f(x, y)}{\partial y}
\]

are defined and continuous on a rectangle \(R\) that contains \((a, b)\) in its interior. Then for some open interval \(I\) containing \(a\), there is a unique solution \(y(x)\) defined for all \(x \in I\).
\[ \frac{dy}{dx} = -\sqrt{1-y^2} \]

Here \[ f(u; \eta) = -\sqrt{1-\eta^2} \]

\[ \frac{\partial f}{\partial \eta} = -\frac{1}{2} \frac{-2\eta}{\sqrt{1-\eta^2}} = \frac{\eta}{\sqrt{1-\eta^2}} \]

Note: def \( \eta = \pm 1 \).