8.3 Spectral Decomposition Methods

Goal: Given an $n \times n$ matrix $A$, find a fast way to compute the matrix $e^{At}$ that comes up in the solution to $\dot{X} = AX$.

Motivation ($n=3$)

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

So,

$$e^{At} = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{5t} \end{pmatrix}$$

Eigenvalues of $A$ are

$$\lambda_1 = 2, \quad \lambda_2 = 3, \quad \lambda_3 = 5$$
Define

\[ P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]

Observe

\[ P_1 + P_2 + P_3 = I \quad (1) \]

\[ P_i \cdot P_j = 0 \quad \text{if} \quad i \neq j \quad (1 \leq i, j \leq 3) \]

\[ P_i^2 = P_i \quad (1 \leq i \leq 3) \]

\[ A = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 \quad (3) \]

Define Kronecker delta

\[ \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]

Then

\[ P_i P_j = \delta_{ij} P_i \quad (1 \leq i, j \leq 3) \quad (2) \]

We now compute \( e^{At} \) using only (1), (2), (3)
Recall
\[ e^{At} = I + At + \frac{A^2 t^2}{2} + \frac{A^3 t^3}{3!} + \ldots \]

Find \( A^2 \) in terms of \( P_1, P_2, P_3 \)

\[ A^2 = (\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3)(\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3) \]
\[ = \lambda_1^2 P_1 + \lambda_2^2 P_2 + \lambda_3^2 P_3 \]

use (2)

Similarly

\[ A^3 = (\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3)^3 \]
\[ = \lambda_1^3 P_1 + \lambda_2^3 P_2 + \lambda_3^3 P_3 \]

etc.

So

\[ e^{At} = I + At + \frac{A^2 t^2}{2} + \frac{A^3 t^3}{3!} + \ldots \]
\[ = P_1 \left( 1 + \lambda_1 t + \frac{\lambda_1^2 t^2}{2} + \frac{\lambda_1^3 t^3}{3!} + \ldots \right) \]
\[ + P_2 \left( 1 + \lambda_2 t + \frac{\lambda_2^2 t^2}{2} + \frac{\lambda_2^3 t^3}{3!} + \ldots \right) \]
\[ + P_3 \left( 1 + \lambda_3 t + \frac{\lambda_3^2 t^2}{2} + \frac{\lambda_3^3 t^3}{3!} + \ldots \right) \]
\[ = P_1 e^{\lambda_1 t} + P_2 e^{\lambda_2 t} + P_3 e^{\lambda_3 t} \]
Find $P_1, P_2, P_3$ in terms of $A$

Claim

$$P_1 = \frac{(A - \lambda_2 I)(A - \lambda_3 I)}{(-\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}$$

$$P_2 = \frac{(A - \lambda_1 I)(A - \lambda_3 I)}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)}$$

$$P_3 = \frac{(A - \lambda_1 I)(A - \lambda_2 I)}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}$$

Check:

$$RHS = \frac{(\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 - \lambda_2 (P_1 + P_2 + P_3))(\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 - \lambda_3 (P_1 + P_2 + P_3))}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}$$

$$= \frac{((\lambda_1 - \lambda_2)P_1 + (\lambda_2 - \lambda_3)P_3)(\lambda_1 - \lambda_2)P_1 + (\lambda_2 - \lambda_3)P_3}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}$$

$$= \frac{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)P_1}{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_3)} = P_1$$
The above claim says

\[ p_i = \prod_{\lambda \neq 0} \frac{A - \lambda I}{\lambda_i - \lambda} \]

Conclusion

\[ e^{At} = e^{\lambda_1 t} p_1 + e^{\lambda_2 t} p_2 + e^{\lambda_3 t} p_3 \]

where

\[ p_i = \prod_{\lambda \neq 0} \frac{A - \lambda I}{\lambda_i - \lambda} \]

More generally ...
LEM \[ \text{Let } A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \text{a } 3 \times 3 \text{ matrix} \]

with 3 distinct eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \).

Define
\[ p_i = \prod_{1 \leq j \neq i \leq 3} \frac{A - \lambda_j I}{\lambda_i - \lambda_j} \]

then
\[ p_1 + p_2 + p_3 = I \]

\[ p_i p_j = \delta_{ij} p_i \quad (1 \leq i, j \leq 3) \]

\[ A = \lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 \]

Moreover
\[ e^{At} = e^{\lambda_1 t} p_1 + e^{\lambda_2 t} p_2 + e^{\lambda_3 t} p_3 \]

pf: Define
\[ D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \]
there exists an invertible matrix $P$ such that

$$A = PDP^{-1}$$

The present theorem is true for $D$. In resulting equations, multi each term on left by $P$ and right by $P^{-1}$. Resulting equations show present theorem holds for $A$ as well. $\ddagger$

More generally ---
**Thm**

Given non-zero matrix $A$ with

distinct eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

Define

$$p_i = \prod_{(i \neq j \in \mathbb{N})} \frac{A - \lambda_j I}{\lambda_i - \lambda_j}$$

(1 to n)

"Projection matrix $p_i$ of $\lambda_i$"

Then

$$p_1 + p_2 + \cdots + p_n = I$$

$$p_i p_j = \delta_{ij} p_i$$

(1 to n)

$$A = \lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_n p_n$$

Moreover

$$e^{At} = e^{\lambda_1 t} p_1 + e^{\lambda_2 t} p_2 + \cdots + e^{\lambda_n t} p_n$$

pf

Sim to above LEM
Ex  Apply above him to

\[
A = \begin{bmatrix}
4 & 2 \\
3 & 4
\end{bmatrix}
\]

Sol  Find eigenvalues of A

\[
| A - \lambda I | = \begin{vmatrix}
4 - \lambda & 2 \\
3 & 4 - \lambda
\end{vmatrix}
\]

\[
= (4 - \lambda)(4 - \lambda) - 12 \\
= \lambda^2 - 8\lambda - 4 \\
= (\lambda - 5)(\lambda + 2)
\]

Eigenvalues of A are

\[
\lambda_1 = 5, \quad \lambda_2 = -2
\]

Find \( p_1 \), \( p_2 \)
\[ P_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2} = \frac{A + 2 I}{7} = \frac{1}{4} \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \]

\[ P_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1} = \frac{A - 5 I}{-7} = \frac{1}{4} \begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \]

Check: \[ P_1 + P_2 = I \]
\[ p_1^2 = p_1 \]
\[ p_1 p_2 = 0 \]
\[ p_2 p_1 = 0 \]
\[ p_2^2 = p_2 \]
\[ A = 5p_1 - 2p_2 \]
\[
\begin{bmatrix}
4 & 2 \\
3 & -1
\end{bmatrix}
= \frac{5}{4}
\begin{bmatrix}
6 & 2 \\
3 & 1
\end{bmatrix}
- \frac{2}{7}
\begin{bmatrix}
1 & -2 \\
-3 & 6
\end{bmatrix}
\]
\[
e^{At} = e^{\lambda_1 t} p_1 + e^{\lambda_2 t} p_2
\]
\[
= e^{5t} \frac{1}{4}
\begin{bmatrix}
6 & 2 \\
3 & 1
\end{bmatrix}
+ e^{-2t} \frac{1}{7}
\begin{bmatrix}
1 & -2 \\
-3 & 6
\end{bmatrix}
\]
Ex

Apply to

\[ A = \begin{bmatrix}
  3 & -4 & 16 \\
  0 & 6 & -15 \\
  0 & 2 & -5 
\end{bmatrix} \]

Sol

Find eigenvalues of \( A \) (skip detail)

\[ \lambda_1 = 0 \quad \lambda_2 = 1 \quad \lambda_3 = 3 \]

Find \( p_1, p_2, p_3 \)

\[ p_1 = \frac{(A - 1I)(A - 3I)}{(0 - 1)(0 - 3)} \]

\[ = \begin{bmatrix}
  0 & 4 & -12 \\
  0 & -5 & 15 \\
  0 & -2 & 6 
\end{bmatrix} \]

\[ p_2 = \frac{(A - 0I)(A - 3I)}{(1 - 0)(1 - 3)} \]

\[ = \begin{bmatrix}
  0 & -4 & 16 \\
  0 & 6 & -15 \\
  0 & 2 & -5 
\end{bmatrix} \]
\[ P_3 = \frac{(A - \sigma I)(A - I)}{(3 - \sigma) (3 - 1)} \]

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

One checks

\[ P_1 + P_2 + P_3 = I \]

\[ P_i P_j = \delta_{ij} P_i \]

\[ 0 P_1 + 1 P_2 + 3 P_3 = A \]

\[ e^{At} = e^{o_1 t} P_1 + e^{t} P_2 + e^{3t} P_3 \]