We now shift topics from differential equations to linear algebra. Later we will apply linear algebra to differential equations.

Ex. Find the solution set for

\[
\begin{align*}
3x + 2y &= 9 \\
x - y &= 8
\end{align*}
\]

Here \(x, y\) are the "unknowns" or "variables".

A solution to (*) is an ordered pair \((x, y)\) of real numbers that makes each equation true.

The solution set is the set of all solutions.

The equations (*) are linear in the variables \(x, y\). Such an equation has the form

\[ax + by = c, \quad a, b, c \in \mathbb{R}\]
terms such as:

\[
x^2, \quad \sqrt{x}, \quad xy, \quad \frac{1}{x} \\
\sin x, \quad e^x, \quad \text{etc.}
\]

or forbidden

Solve \( x \):

\[
\begin{align*}
x &= y + 8 \\
3(y + 8) + 2y &= 9 \\
5y + 24 &= 9 \\
5y &= -15 \\
y &= -3 \\
x &= -3 + 8 \\
x &= 5
\end{align*}
\]

\( x = 5, \ y = -3 \) is unique solution.
Solution for $3x + 2y = 9$ is a line.
Solution for $x - y = 8$ is a line.
Solution at $x$ is where the lines intersect.
Given a system of 2 linear equations on \(x, y\), there are 3 possibilities for the solution set:

<table>
<thead>
<tr>
<th>Case</th>
<th>Example</th>
<th>Graph</th>
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</thead>
<tbody>
<tr>
<td>Unique sol</td>
<td>(3x + 2y = 9)</td>
<td>![parallel lines]</td>
</tr>
<tr>
<td></td>
<td>(x - y = 8)</td>
<td></td>
</tr>
<tr>
<td>No soli</td>
<td>(x + y = 1)</td>
<td>![lines coincide]</td>
</tr>
<tr>
<td></td>
<td>(2x + 2y = 3)</td>
<td></td>
</tr>
<tr>
<td>oo many sols</td>
<td>(x + y = 1)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(2x + 2y = 2)</td>
<td></td>
</tr>
</tbody>
</table>
Ex Consider the linear system in the variables $x, y, z$:

$$
\begin{align*}
2x + 7y + 3z &= 11 \\
x + 3y + 2z &= 2 \\
3x + 7y + 9z &= -12
\end{align*}
$$

Find the solution set.

**Strategy**

We employ three types of moves to transform the system into a simpler linear system that has the same solution set.

(i) For some equation, multiply each side by the same non-zero constant

(ii) Interchange two equations

(iii) Add a constant multiple of some equation to another equation

"Elementary operations"
Using the elementary operations we try to put \( x \) in the form

\[-x + -y + -z = -
\]
\[-y + -z = -
\]
\[-z = -
\]

"triangular form"

We then "backsolve" to find \( z \), then \( y \), then \( x \).

I interchange egs one and two:

\[x + 3y + 2z = 2 \]
\[2x + 7y + 3z = 11\]
\[3x + 7y + 9z = -12\]
II Replace eq 2 by eq 2 - 2 eq 1:

\[ x + 3y + 2z = 2 \]
\[ y - z = 7 \]
\[ 3x + 7y + 9z = -12 \]

III Replace eq 3 by eq 3 - 3 eq 1:

\[ x + 3y + 2z = 2 \]
\[ y - z = 7 \]
\[ -2y + 3z = -18 \]

III Replace eq 3 by eq 3 + 2 eq 2:

\[ x + 3y + 2z = 2 \]
\[ y - z = 7 \]
\[ z = -4 \]

(triangular form)
Backsolve

\[ z = -y \]
\[ y = x + z \]
\[ = x - y \]
\[ = 3 \]

\[ x = 2 - 3y - 2z \]
\[ = 2 - 9 + 8 \]
\[ = 1 \]

Unique sol to x is

\[ x = 1, \quad y = 3, \quad z = -y \]

Double Check:

\[ 2 \cdot 1 + 7 \cdot 3 + 3(-y) = 11 \]
\[ 1 \cdot 1 + 3 \cdot 3 + 2(-y) = 2 \]
\[ 3 \cdot 1 + 7 \cdot 3 + 9(-y) = -12 \]
Ex. Find the solution set for the linear system:

\[
\begin{align*}
  x - 3y + 2z &= 6 \\
  x + 4y - z &= 9 \\
  5x + 6y + z &= 20
\end{align*}
\]

Sol. Apply elem. ops to put in triangular form:

I. Replace \( e_2 \) by \( e_2 - e_1 \):

\[
\begin{align*}
  x - 3y + 2z &= 6 \\
  7y - 3z &= -2 \\
  5x + 6y + z &= 20
\end{align*}
\]

II. Replace \( e_3 \) by \( e_3 - 5e_1 \):

\[
\begin{align*}
  x - 3y + 2z &= 6 \\
  7y - 3z &= -2 \\
  21y - 9z &= -10
\end{align*}
\]
III Replace eq 3 by eq 3 - 3eq 2:

\[ x - 3y + 2z = 6 \]
\[ 7y - 3z = -2 \]
\[ 0 = -4 \]

The last equation shows

No sol \ (ie Solution set is empty)
Ex. Find the solution set for the linear system

\[ \begin{align*}
    x + y - 2z &= 5 \\
    3x + y + 3z &= 11 \\
    4x + y + 5z &= 14
\end{align*} \]

Sol. Apply elem ops to put \( x \) in bray form

I. Replace eq2 by eq2 - 3 eq1:

\[ \begin{align*}
    x + y - 2z &= 5 \\
    -2y + 6z &= -4 \\
    4x + y + 5z &= 14
\end{align*} \]

II. Replace eq3 by eq3 - 4 eq1:

\[ \begin{align*}
    x + y - 2z &= 5 \\
    -3y + 6z &= -4 \\
    -3y + 9z &= -6
\end{align*} \]
III
Replace eq 2 by \(-\frac{1}{2}\) eq 2:

\[
\begin{align*}
    x + y - z &= 5 \\
    y - 3z &= 2 \\
    -3y + 9z &= -6
\end{align*}
\]

IV
Replace eq 3 by \(-eq 3 + 3eq 2:

\[
\begin{align*}
    x + y - z &= 5 \\
    y - 3z &= 2 \\
    0 &= 0
\end{align*}
\]

(triangular form)

Backsolve:

no constraint on \(z\), no write

\[
\begin{align*}
    z &= t \\
    y &= 2 + 3z \quad \text{free} \\
    &= 2 + 3t
\end{align*}
\]

\[
\begin{align*}
    x &= 5 - y + z \\
    &= 5 - (2 + 3t) + t \\
    &= 3 - 2t
\end{align*}
\]
Sol: set is

\[ x = 3 - 2t, \quad y = 2 + 3t, \quad z = t \quad \text{free} \]

Double check:

1. \[ 1(3 - 2t) + 1(2 + 3t) - t = 5 \quad \checkmark \]
2. \[ 3(3 - 2t) + 1(2 + 3t) + 3t = 11 \quad \checkmark \]
3. \[ 4(3 - 2t) + 1(2 + 3t) + 5t = 14 \quad \checkmark \]
Consider the function
\[ y = e^{5x} \]

Obs
\[ y' = 5e^{5x} = 5y \]

So
\[ y'' = 5y' = 25y \]

So \( y = e^{5x} \) is a particular solution to
\[ y'' - 25y = 0 \]

Consider the function
\[ y = e^{-5x} \]

Here
\[ y' = -5e^{-5x} = -5y \]

So
\[ y'' = -5y' = 25y \]

So \( y = e^{-5x} \) is another particular solution to
\[ y'' - 25y = 0 \]
As we will see, the general solution to
\[ y'' - 25y = 0 \]
is
\[ y = Ae^{5x} + Be^{-5x}, \quad A, B \text{ constants}. \]

**Ex.** Solve the initial value problem
\[ y'' - 25y = 0, \]
\[ y(0) = 10, \quad y'(0) = 20. \]

**Sol.**
\[ y = Ae^{5x} + Be^{-5x}, \quad A, B = \text{constants} \]

Find \( A, B \)

**obs.**
\[ y' = 5Ae^{5x} - 5Be^{-5x} \]

**Require**
\[ 10 = y(0) = A + B \]
\[ 20 = y'(0) = 5A - 5B \]

Solve the linear system in variables \( A, B \):
\[ A + B = 10 \]
\[ 5A - 5B = 20 \]

\[ A = 7, \quad B = 3 \]
is unique sol

\[ y = 7e^{5x} + 3e^{-5x} \]