We classified the LS in 16.222.

We now relate LS to the polynomials from terminating branch of the Askey scheme.

Let $\mathcal{E} = (A_i, E_i, A_i^x, E_i^x)$ denote LS on $V$

Let $L = (a_i, o_i, v_i, \phi_i)$

In 19.3 we saw some polys $P_i, \overline{P}_i, \mathbb{C} = \mathbb{F}[x]$

5.6

$E_i^x V = p_i^x(A) E_i^x V \quad o \in \mathbb{N}$

For $o \leq i \leq n$ $p_i$ defined up to nonzero scalar multi

work $p_i^{\text{monic}}$

Def $p_{\text{min}} = (\text{monic}) \min_{i \leq n} \text{poly of } A$

It is convenient to work with a certain normalization

$\psi$ is called $u_i, v_i$
The standard basis

Recall from L193 that

\[ \mathbb{E}_0 \text{ is normalizing} \]

So

\[ \mathbb{E}_0 \text{ is normalizing} \]

So for \( \alpha \in \mathbb{C} \):

\[ \mathbb{E}_i^x \mathbb{E}_0 \neq 0 \]

So

\[ \mathbb{E}_i^x \mathbb{E}_0 \mathbb{V} \neq 0 \]

So

\[ \mathbb{E}_i^x \mathbb{E}_0 \mathbb{V} = \mathbb{E}_i^x \mathbb{V} \]

This gives:

LEM 234

Given \( \alpha \mathbb{E}_0 \mathbb{V} \), \( \mathbb{E}_i^x \mathbb{U} \) is known \( \mathbb{E}_i^x \mathbb{V} \)

Moreover

\[ \{ \mathbb{E}_i^x \mathbb{U} \}_{i=0}^\alpha \text{ is known } \mathbb{V} \]

pf

\[ \square \]
LEM 235  Given vectors \[ \{ w_i \}^{\infty}_{i=0} \] in \( V \), not all \( 0 \).

Then \( \{ w_i \}^{\infty}_{i=0} \) is a \( \mathbb{F} \)-stand basis for \( V \) iff both

(i) \[ w_i \in E_i^* \] \( \forall i \in \mathbb{N} \)

(ii) \[ \sum_{i=0}^{\infty} w_i \in E_0 \]

pf \text{ ex}

LEM 236  Given basis \( \{ w_i \}^{\infty}_{i=0} \) for \( V \)

Let \( B = \text{mat in } \mathbb{M}_{n\times n}(\mathbb{F}) \text{ with } w_i \text{ on } A \) and \( \{ w_i \}^{\infty}_{i=0} \)

\[ B^* = \cdots \]

then \( \{ w_i \}^{\infty}_{i=0} \) is \( \mathbb{F} \)-st basis iff both

(i) \( B \) has cost row sum \( \theta_0 \)

(ii) \( B^* = \text{diag} ( \theta_i )^{\infty}_{i=0} \)

pf \text{ ex}
Def 2.3.7 \( \forall X \in \text{End} V \) let \( X^b \) denote matrix \( b \) in \( \text{Mat}_n(\mathbb{F}) \) that reps \( X \) wrt \( \mathcal{E} \)-stand base for \( V \).

\[
\begin{align*}
\text{obs.} & \quad \text{End} V \to \text{Mat}_n(\mathbb{F}) \\
& \quad b : X \to X^b \\
& \quad \text{is iso of } \mathbb{F}-\text{algebra.}
\end{align*}
\]

By constr and L2.36
\[
A^b = \text{vec} \text{ tridiag with const row sum } \theta_0.
\]
\[
A^b = \text{diag} (\theta^b) \circ_0
\]
\[
E^b = \begin{pmatrix}
0 & i \\\ni & 0
\end{pmatrix}
\]

Write
\[
A^b = \\
\begin{pmatrix}
\alpha & \beta & 0 \\
\gamma & \delta & 0 & \ddots \\
0 & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}
\]

\[F_n \circ \circ \circ \]

\[
a_i = \text{tr } A^b E_i^b
\]

is same as \( \text{def } 18.4 \).

\[
\begin{align*}
\text{obs.} & \quad c_i + b_i = \theta_0, \quad \text{for } i \in \mathbb{N} \\
& \quad c_0 = 0, \quad b_N = 0
\end{align*}
\]
Def \( \{v_i\}_{i=0}^N \) in \( \mathbb{F}[x] \) by

\[
x v_i = b_i v_i + a_{i+1} v_{i+1} + c_i v_{i-1}, \quad 0 \leq i \leq N-1
\]

\[
v_0 = 1, \quad v_N = 0
\]

For \( 0 \leq i \leq N \), let \( v_i \) have degree \( i \)

leading coeff is \( \frac{1}{c_1 \cdots c_i} \)

Let \( \{w_i\}_{i=0}^N \) denote \( \mathbb{F} - \text{st basis for } V \)

By casework \( v_i(A) w_0 = w_i \quad \forall \leq i \leq N \)

So \( v_i \in \mathbb{F} p_i \quad \forall \leq i \leq N \)

Comparing leading coeffs

\[
v_i = \frac{p_i}{c_1 \cdots c_i} \quad \forall \leq i \leq N
\]

Def \( \{u_i\}_{i=0}^N \) in \( \mathbb{F}[x] \) by

\[
x u_i = c_i u_{i-1} + a_{i+1} u_i + b_i u_{i+1} \quad \forall \leq i \leq N-1
\]

\[
u_0 = 1, \quad u_1 = 0
\]

For \( 0 \leq i \leq N \), let \( u_i \) have degree \( i \)

leading coeff is \( \frac{1}{b_0 \cdots b_i} \)

One choice

\[
v_i = u_i k_i \quad \forall \leq i \leq N
\]

where \( k_i = \frac{b_0 b_1 \cdots b_i}{c_1 c_2 \cdots c_i} \)
\[ u_i \in \mathbb{F} \quad p_i \quad 0 \leq i \leq N \]

So,
\[ u_i = \frac{p_i}{\prod_{j \neq i} \lambda_j} \quad 0 \leq i \leq N \]

Using the def of \( \{u_i\}_{i=0}^N \) and \( X \), we have
\[ u_i(\infty) = 1 \quad 0 \leq i \leq N \]

So,
\[ p_i(\infty) = \prod_{j \neq i} \lambda_j \quad 0 \leq i \leq N \]
Theorem 232  \[ \forall \alpha \leq \pi \in \mathbb{N} \]

\[ u_i^\alpha = \sum_{h=0}^{\alpha} \frac{(x-a_0)(x-a_1) \cdots (x-a_{h-1})(\alpha - a_h)(\alpha - a_{h+1}) \cdots (\alpha - a_N)}{\Psi_h, \Psi_{h+1} - \Psi_N} \]

Proof: Since \( u_i \) is deg \( \alpha \)

\[ u_i = \sum_{h=0}^{\alpha} \lambda_h T_h \]

Show

\[ \lambda_h = \frac{T_h(\alpha)}{\Psi_h, \Psi_{h+1} - \Psi_N} \quad 0 \leq h \leq \alpha \]

To do this, show

\[ \lambda_0 = 1 \]

\[ \lambda_h (\alpha - \alpha_h) = \lambda_0 \Psi_{h+1} \quad 0 \leq h \leq \alpha - 1 \]

To get \( \lambda_0 = 1 \) apply \( \lambda_i \) \( h = 0 \)

\[ 1 = u_i(\alpha) \]

\[ = \sum_{h=0}^{\alpha} \lambda_h T_h(\alpha) \]

\[ = \begin{cases} 1 & \text{if } h=0 \\ 0 & \text{if } h>0 \end{cases} \]

\[ = \lambda_0 \]

Proof: \( Fix \alpha \neq 0 \in E^\alpha_0 V \)

Recall \( u_i(A) \in E^\alpha_0 V \)

\( E^\alpha_0 V \) is subspace of \( A^\alpha \) equal \( \Phi_i \alpha \)
\[(A^*-\theta^*I)u; (A\nu) = 0\]

We saw earlier that the basis:
\[
\{ T_h(A)\nu \}_{h=0}^N
\]

matrices for \(A^*\) is
\[
\begin{pmatrix}
\theta_0^* & \varphi_1 & 0 & \cdots & 0 \\
0 & \ddots & \varphi_N & & \\
0 & & \ddots & \varphi_N \\
0 & & & \theta_N^*
\end{pmatrix}
\]

So
\[
0 = (A^*-\theta^*I)u; (A\nu)
\]
\[
= (A^*-\theta^*I) \sum_{h=0}^N \lambda_h T_h(A)\nu
\]
\[
= \sum_{h=0}^N \lambda_h \left( (\theta_h^*-\theta_h^*) T_h(A)\nu + \varphi_h T_h(A)\nu \right)
\]
\[
= \sum_{h=0}^N T_h(A)\nu \left( \lambda_h (\varphi_h) - \lambda_h (\theta_h^*-\theta_h^*) \right)
\]

must be 0.
Lem 239. For $0 \leq i \leq N$

$$p_i(\theta_0) = \frac{\psi_i, \psi_2, \ldots, \psi_i}{r_i^x(\theta_i^x)}$$

pf. We saw the leading coeff of $u_i$ is

$$\frac{1}{b_0 b_1 \ldots b_i} = \frac{1}{p_i(\theta_0)}$$

By th 238, the leading coeff of $u_i$ is

$$\frac{r_i^x(\theta_i^x)}{\psi_i, \psi_2, \ldots, \psi_i}$$
\[ f_{hm} \quad \text{For} \quad 0 \leq i \leq N \]

\[ \rho_i = \sum_{h=0}^{\xi} \frac{\psi_i \psi_{i+1} \ldots \psi_{i+\xi}}{\psi_i \psi_{i+1} \ldots \psi_h} \frac{T_h(\theta_i^x)}{T_i(\theta_i^z)} \tau_i \]

\[ \rho_i = \rho_i(\theta_i) \quad \uparrow \quad \uparrow \]

use th. 23.8

\[ \square \]
For $0 < i < N$

$$p_i^* = \sum_{k=0}^{i} \frac{\phi_i \phi_k \cdots \phi_k}{\phi_i \phi_k \cdots \phi_k} \frac{\tau_i^* (\theta_i^*)}{\tau_i^* (\theta_i^*)} \tau_i^*$$

pf

Obs $p_i^*$ is inv if we replace $\Xi$ by $\Xi^\psi$.

By def $p_i^*$ is monic deg $i$ and

$$p_i(A) E_0^{x_N} = E_i^{x_N}$$

For $0 < i < N$

$$E_i^x (\Xi^\psi) = E_0^{x_i}$$

Now apply Thm 240 to $\Xi^\psi$ and use

$$\Xi \rightarrow \Xi^\psi$$

$$\psi_i \rightarrow \phi_i$$

$$\tau_i \rightarrow \eta_i$$

$$\tau_i^* \rightarrow \tau_i^*$$

$$\theta_i^* \rightarrow \theta_i^*$$

\[\square\]
\[ u_i = \frac{\phi_1 \cdots \phi_i}{\psi_1 \cdots \psi_i} \sum_{h=0}^{k} \frac{(x-\omega_n)(x-\omega_m) \cdots (h-\omega_m-\omega_n)(\omega_i^h-\omega_j^h)(\omega_i^h-\omega_k^h) \cdots (\omega_i^h-\omega_m^h)}{\phi_1 \phi_2 \cdots \phi_h} \]

pf

\[ u_i = \frac{\rho_i}{\rho_i(\omega)} \left\langle \text{use } \text{A}241 \right\rangle \]

\[ \rho_i(\omega) \left\langle \text{use } \text{L}239 \right\rangle \]

\[ \square \]
For our LS \( \mathcal{F} \) we defined some polynomials \( \{ u_i \}_{i=0}^N \).

Let \( \{ u_i^* \}_{i=0}^N \) denote corresponding polynomials for \( \mathcal{F}^* \).

**Theorem 2.43**

\[
F_n \quad 0 \leq i, j \leq N
\]

\[
u_i(\varphi_j) = u_i^*(\varphi_j^n)
\]

\[\text{Ask eq. - eq. 4.16m}
\]

\[\text{duality}
\]

**Proof**

By eq. 2.38

\[
u_i(\varphi_j) = \sum_{h=0}^d \frac{T_h(\varphi_j) T_h^*(\varphi_j^n)}{\varphi_i - \varphi_h}
\]

Applying eq. 2.4 to \( \mathcal{F}^* \)

\[
u_i^*(\varphi_j^n) = \sum_{h=0}^d \frac{T_h(\varphi_j) T_h^*(\varphi_j^n)}{\varphi_i^* - \varphi_h^*}
\]

But \( \varphi_h^* = \varphi_h \) \( \forall h \leq N \)

by Prop 2.21

Result follows.
We now link the polynomials $\{u_i\}_{i=0}^N$ to the Askey scheme.

start with the case of Krawtchouk type.

**LEM 24.4** Assume

$$\theta_i = i$$

$$\theta_i^N = i$$

then for $\theta = 0$ we have $\theta > N$ by PA2.

$$\varphi_i = \frac{\sum_{h=0}^{i-1} \theta_h - \theta_{N-h}}{\theta_0 - \theta_N}$$

$$\varphi_i = \frac{\sum_{h=0}^{i-1} h - (N-h)}{0-N}$$

$$= \frac{i \cdot (N-i\bar{m})}{N}$$

**pf**

For $i \leq i \in N$

$$\sum_{h=0}^{i-1} \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} = \sum_{h=0}^{i-2} \frac{h - (N-h)}{0-N}$$

$$= \frac{i \cdot (N-i\bar{m})}{N}$$

**Def**

$$p = -\frac{\varphi_i}{N}$$

so $\varphi_i = -pN$

By PA4, $p \leq i \in N$

$$\phi_i = \varphi_i \sum_{h=0}^{i-2} \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} + \left( \theta_i^* - \theta_0^2 \right) \frac{\theta_{N-h} - \theta_0}{\theta_0 - \theta_N}$$

$$= -pN \cdot \frac{i \cdot (N-i\bar{m})}{N} + i \cdot \frac{i \cdot (N-i\bar{m})}{N}$$

$$= (1-p) i \cdot (N-i\bar{m})$$

so $\phi_i = (1-p)N$
By PA 3. \[ \lambda \in i \mathbb{S}_N \]

\[ \Phi_c = \phi \sum_{h=0}^{2} \frac{\theta_h - \theta_{N-h}}{\theta_0 - \theta_N} + \frac{(\theta_i^* - \theta_0^*) \sqrt{\theta_{17} - \theta_N}}{N} \]

\[ = (1-p)N \cdot \frac{i(N-iM)}{N} + i' \cdot (i'-1-N) \]

\[ = -p \lambda' (N-iM) \]
Assume

\[ \Theta_i = i, \quad \Theta^*_i = i \quad \text{for } i \in \mathbb{Z}_N \]

Then

\[ u_i(x) = K_i^s(x; p, N) \quad \text{for } i \in \mathbb{Z}_N \]

where \( p \) is from L244

pf In th 238 recall \( u_i(x) \) using

\[ \Theta_i = i \]
\[ \Theta^*_i = i \]
\[ \eta^*_i = -p i^*(N - i + 1) \]

so

\[ u_i(x) = 2F_1 \left( \frac{-i - x}{N}; \frac{1}{p} \right) \]

\[ = K_i^s(x; p, N) \]

Note: Ref to th 245,

Since \( \Theta_i = \Theta^*_i \), \( i \in \mathbb{Z}_N \)

So \( u_i(x) = u_i^*(x) \) \( i \in \mathbb{Z}_N \)

So by duality th 243 asserts

\[ u_i(x) = u_{i*}(x) \quad i \in \mathbb{Z}_N \]

\[ K_i^s(x; p, N) = K_i^s(x_2; p, N) \]

\[ = 2F_1 \left( \frac{-i - x}{N}; \frac{1}{p} \right) \]
In the handout we list all the parameter arrays on $F$.

For each array $(\theta; \theta^x; \psi; \psi^x)$ we give

$$u_{\psi}(\theta) = \varepsilon_{\psi^x}$$

The resulting formula shows that $\{u_{\psi} \}$ are from Askey scheme.

Also, every poly sequence from term branch of Askey scheme is realized as $\{u_{\psi} \}$ for some PA.
Theorem 34.14  [51, Section 10] Assume $\mathbb{K}$ is algebraically closed. Let $q$ denote a nonzero scalar in $\mathbb{K}$ that is not a root of unity. Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $A, A^*$ denote a tridiagonal pair on $V$ that has $q$-geometric type. Then there exists an irreducible $\mathbb{K}_q$-module structure on $V$ such that $A$ acts as a scalar multiple of $x_{01}$ and $A^*$ acts as a scalar multiple of $x_{23}$. Conversely, let $V$ denote a finite dimensional irreducible $\mathbb{K}_q$-module. Then the generators $x_{01}, x_{23}$ act on $V$ as a tridiagonal pair of $q$-geometric type.

We end this section with a conjecture.

Conjecture 34.15 Assume $\mathbb{K}$ is algebraically closed. Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension and let $A, A^*$ denote a tridiagonal pair on $V$. To avoid degenerate situations we assume $q$ is not a root of unity, where $\beta = q^2 + q^{-2}$, and where $\beta$ is from Theorem 34.8. Then referring to Definition 34.12, there exists an irreducible $\mathbb{K}_q$-module structure on $V$ such that $A$ acts as a linear combination of $x_{01}, x_{12}, I$ and $A^*$ acts as a linear combination of $x_{23}, x_{30}, I$.

35  Appendix: List of parameter arrays

In this section we display all the parameter arrays over $\mathbb{K}$. We will use the following notation.

Definition 35.1 Let $p = (\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ denote a parameter array over $\mathbb{K}$. For $0 \leq i \leq d$ we let $u_i$ denote the following polynomial in $\mathbb{K}[\lambda]$.

$$u_i = \sum_{n=0}^{i} \frac{(\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{n-1})(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_1^*) \cdots (\theta_i^* - \theta_{n-1}^*)}{\varphi_1 \varphi_2 \cdots \varphi_n}.$$  \hspace{1cm} (134)

We call $u_0, u_1, \ldots, u_d$ the polynomials that correspond to $p$.

We now display all the parameter arrays over $\mathbb{K}$. For each displayed array $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ we present $u_i(\theta_j)$ for $0 \leq i, j \leq d$, where $u_0, u_1, \ldots, u_d$ are the corresponding polynomials. Our presentation is organized as follows. In each of Example 35.2–35.14 below we give a family of parameter arrays over $\mathbb{K}$. In Theorem 35.15 we show every parameter array over $\mathbb{K}$ is contained in at least one of these families.

In each of Example 35.2–35.14 below the following implicit assumptions apply: $d$ denotes a nonnegative integer, the scalars $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ are contained in $\mathbb{K}$, and the scalars $q, h, h^* \ldots$ are contained in the algebraic closure of $\mathbb{K}$.

Example 35.2 ($q$-Racah) Assume

$$\theta_i = \theta_0 + h(1 - q^i)(1 - sq^{i+1})q^{-i},$$ \hspace{1cm} (135)

$$\theta_i^* = \theta_0^* + h^*(1 - q^i)(1 - s^*q^{i+1})q^{-i}$$ \hspace{1cm} (136)

for $0 \leq i \leq d$ and

$$\varphi_i = hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(1 - r_1q^i)(1 - r_2q^i),$$ \hspace{1cm} (137)

$$\phi_i = hh^*q^{1-2i}(1 - q^i)(1 - q^{i-d-1})(r_1 - s^*q^i)(r_2 - s^*q^i)/s^*$$ \hspace{1cm} (138)
for \(1 \leq i \leq d\). Assume \(h, h^*, q, s, s^*, r_1, r_2\) are nonzero and \(r_1 r_2 = ss^* q^{d+1}\). Assume none of 
\(q^i, r_1 q^i, r_2 q^i, s^* q^i / r_1, s^* q^i / r_2\) is equal to 1 for \(1 \leq i \leq d\) and that neither of \(sq^i, s^* q^i\) is equal to 1 for \(2 \leq i \leq 2d\). Then \((\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)\) is a parameter array over \(\mathbb{K}\). The corresponding polynomials \(u_i\) satisfy

\[
u_i(\theta_j) = 4\phi_3\left(\begin{array}{c} q^{-i}, s^* q^{i+1}, q^{-j}, sq^{i+1} \\
r_1 q, r_2 q, q^{-d} \end{array} \right| q, q)\]

for \(0 \leq i, j \leq d\). These \(u_i\) are the \(q\)-Racah polynomials.

**Example 35.3 (q-Hahn)** Assume

\[
\theta_i = \theta_0 + h(1 - q^i)q^{-i}, \\
\theta^*_i = \theta_0^* + h^*(1 - q^i)(1 - s^* q^{i+1})q^{-i}
\]

for \(0 \leq i \leq d\) and

\[
\varphi_i = hh^* q^{1-2i}(1 - q^i)(1 - q^{-i-d})(1 - rq^i), \\
\phi_i = -hh^* q^{-1-i}(1 - q^i)(1 - q^{i-d})(r - s^* q^i)
\]

for \(1 \leq i \leq d\). Assume \(h, h^*, q, s^*, r\) are nonzero. Assume none of \(q^i, rq^i, s^* q^i / r\) is equal to 1 for \(1 \leq i \leq d\) and that \(s^* q^i \neq 1\) for \(2 \leq i \leq 2d\). Then the sequence \((\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)\) is a parameter array over \(\mathbb{K}\). The corresponding polynomials \(u_i\) satisfy

\[
u_i(\theta_j) = 3\phi_2\left(\begin{array}{c} q^{-i}, s^* q^{i+1}, q^{-j} \\
rq, q^{-d} \end{array} \right| q, q)\]

for \(0 \leq i, j \leq d\). These \(u_i\) are the \(q\)-Hahn polynomials.

**Example 35.4 (Dual q-Hahn)** Assume

\[
\theta_i = \theta_0 + h(1 - q^i)(1 - sq^{i+1})q^{-i}, \\
\theta^*_i = \theta_0^* + h^*(1 - q^i)q^{-i}
\]

for \(0 \leq i \leq d\) and

\[
\varphi_i = hh^* q^{1-2i}(1 - q^i)(1 - q^{-i-d})(1 - rq^i), \\
\phi_i = hh^* q^{d+2-2i}(1 - q^i)(1 - q^{i-d})(s - rq^{-d})
\]

for \(1 \leq i \leq d\). Assume \(h, h^*, q, r, s\) are nonzero. Assume none of \(q^i, rq^i, sq^i / r\) is equal to 1 for \(1 \leq i \leq d\) and that \(sq^i \neq 1\) for \(2 \leq i \leq 2d\). Then the sequence \((\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)\) is a parameter array over \(\mathbb{K}\). The corresponding polynomials \(u_i\) satisfy

\[
u_i(\theta_j) = 3\phi_2\left(\begin{array}{c} q^{-i}, q^{-j}, sq^{i+1} \\
rq, q^{-d} \end{array} \right| q, q)\]

for \(0 \leq i, j \leq d\). These \(u_i\) are the dual \(q\)-Hahn polynomials.
Example 35.5 (Quantum q-Krawtchouk) Assume

\[
\theta_i = \theta_0 - sq(1 - q^i),
\]

\[
\theta_i^* = \theta_0^* + h^*(1-q^i)q^{-i}
\]

for \(0 \leq i \leq d\) and

\[
\varphi_i = -rh^*q^{1-i}(1-q^i)(1-q^{i-d-1}),
\]

\[
\phi_i = q^*q^{d+2-2i}(1-q^i)(1-q^{i-d-1})(s-rq^{i-d-1})
\]

for \(1 \leq i \leq d\). Assume \(h^*, q, r, s\) are nonzero. Assume neither of \(q^i, sq^i/r\) is equal to 1 for \(1 \leq i \leq d\). Then the sequence \((\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)\) is a parameter array over \(\mathbb{K}\). The corresponding polynomials \(u_i\) satisfy

\[
u_i(\theta_j) = 2\phi_1\left(\begin{array}{cc} q^{-i}, & q^{-j} \\ q^{-d} & \end{array} \right | q, \ sr^{-1}q^{j+1})
\]

for \(0 \leq i, j \leq d\). These \(u_i\) are the quantum q-Krawtchouk polynomials.

Example 35.6 (q-Krawtchouk) Assume

\[
\theta_i = \theta_0 + h(1-q^i)q^{-i},
\]

\[
\theta_i^* = \theta_0^* + h^*(1-q^i)(1-s^*q^{i+1})q^{-i}
\]

for \(0 \leq i \leq d\) and

\[
\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1}),
\]

\[
\phi_i = hh^*s^*q(1-q^i)(1-q^{i-d-1})
\]

for \(1 \leq i \leq d\). Assume \(h, h^*, q, s^*\) are nonzero. Assume \(q^i \neq 1\) for \(1 \leq i \leq d\) and that \(s^*q^i \neq 1\) for \(2 \leq i \leq 2d\). Then the sequence \((\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)\) is a parameter array over \(\mathbb{K}\). The corresponding polynomials \(u_i\) satisfy

\[
u_i(\theta_j) = 3\phi_2\left(\begin{array}{cc} q^{-i}, & s^*q^{i+1}, q^{-j} \\ q^{-d} & \end{array} \right | q, q)
\]

for \(0 \leq i, j \leq d\). These \(u_i\) are the q-Krawtchouk polynomials.

Example 35.7 (Affine q-Krawtchouk) Assume

\[
\theta_i = \theta_0 + h(1-q^i)q^{-i},
\]

\[
\theta_i^* = \theta_0^* + h^*(1-q^i)q^{-i}
\]

for \(0 \leq i \leq d\) and

\[
\varphi_i = hh^*q^{1-2i}(1-q^i)(1-q^{i-d-1})(1-rq^i),
\]

\[
\phi_i = hhh^*rq^{1-i}(1-q^i)(1-q^{i-d-1})
\]
for $1 \leq i \leq d$. Assume $h, h^*, q, r$ are nonzero. Assume neither of $q^i, r q^i$ is equal to 1 for $1 \leq i \leq d$. Then the sequence $(\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials $u_i$ satisfy

$$u_i(\theta_j) = 3\phi_2\left( q^{-i}, 0, q^{-j} \middle| q, q \right).$$

for $0 \leq i, j \leq d$. These $u_i$ are the affine $q$-Krawtchouk polynomials.

**Example 35.8 (Dual $q$-Krawtchouk)** Assume

$$\theta_i = \theta_0 + h(1 - q^i)(1 - sq^{i+1})q^{-i},$$
$$\theta^*_i = \theta^*_0 + h^*(1 - q^i)q^{-i}$$

for $0 \leq i \leq d$ and

$$\varphi_i = hh^* q^{1-2i}(1 - q^i)(1 - q^{i-d-1}),$$
$$\phi_i = hh^* sq^{2i-2}(1 - q^i)(1 - q^{i-d-1})$$

for $1 \leq i \leq d$. Assume $h, h^*, q, s$ are nonzero. Assume $q^i \neq 1$ for $1 \leq i \leq d$ and $sq^i \neq 1$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials $u_i$ satisfy

$$u_i(\theta_j) = 3\phi_2\left( q^{-i}, q^{-j}, sq^{i+1} \middle| q, q \right)$$

for $0 \leq i, j \leq d$. These $u_i$ are the dual $q$-Krawtchouk polynomials.

**Example 35.9 (Racah)** Assume

$$\theta_i = \theta_0 + hi(i + 1 + s),$$
$$\theta^*_i = \theta^*_0 + h^*i(i + 1 + s^*)$$

for $0 \leq i \leq d$ and

$$\varphi_i = hh^*i(i - d - 1)(i + r_1)(i + r_2),$$
$$\phi_i = hh^*i(i - d - 1)(i + s^* - r_1)(i + s^* - r_2)$$

for $1 \leq i \leq d$. Assume $h, h^*$ are nonzero and that $r_1 + r_2 = s + s^* + d + 1$. Assume the characteristic of $\mathbb{K}$ is 0 or a prime greater than $d$. Assume none of $r_1, r_2, s^* - r_1, s^* - r_2$ is equal to $-i$ for $1 \leq i \leq d$ and that neither of $s, s^*$ is equal to $-i$ for $2 \leq i \leq 2d$. Then the sequence $(\theta_i, \theta^*_i, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials $u_i$ satisfy

$$u_i(\theta_j) = _4F_3\left( -i, i + 1 + s^*, -j, j + 1 + s \middle| r_1 + 1, r_2 + 1, -d \right)$$

for $0 \leq i, j \leq d$. These $u_i$ are the Racah polynomials.
Example 35.10 (Hahn) Assume
\[ \theta_i = \theta_0 + si, \]
\[ \theta_i^* = \theta_0^* + h^*i(i+1+s^*) \]
for \(0 \leq i \leq d\) and
\[ \varphi_i = h^*si(i-d-1)(i+r), \]
\[ \phi_i = -h^*si(i-d-1)(i+s^*-r) \]
for \(1 \leq i \leq d\). Assume \(h^*, s\) are nonzero. Assume the characteristic of \(K\) is 0 or a prime greater than \(d\). Assume neither of \(r, s^*-r\) is equal to \(-i\) for \(1 \leq i \leq d\) and that \(s^* \neq -i\) for \(2 \leq i \leq 2d\). Then the sequence \((\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)\) is a parameter array over \(K\). The corresponding polynomials \(u_i\) satisfy
\[ u_i(\theta_j) = _3F_2 \left( \begin{array}{c} -i, i+1+s^*, -j \\ r+1, -d \end{array} \right| 1 \right) \]
for \(0 \leq i, j \leq d\). These \(u_i\) are the Hahn polynomials.

Example 35.11 (Dual Hahn) Assume
\[ \theta_i = \theta_0 + hi(i+1+s), \]
\[ \theta_i^* = \theta_0^* + s^*i \]
for \(0 \leq i \leq d\) and
\[ \varphi_i = hs^*i(i-d-1)(i+r), \]
\[ \phi_i = hs^*i(i-d-1)(i+r-s-d-1) \]
for \(1 \leq i \leq d\). Assume \(h, s^*\) are nonzero. Assume the characteristic of \(K\) is 0 or a prime greater than \(d\). Assume neither of \(r, s - r\) is equal to \(-i\) for \(1 \leq i \leq d\) and that \(s \neq -i\) for \(2 \leq i \leq 2d\). Then the sequence \((\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)\) is a parameter array over \(K\). The corresponding polynomials \(u_i\) satisfy
\[ u_i(\theta_j) = _3F_2 \left( \begin{array}{c} -i, -j, j+1+s \\ r+1, -d \end{array} \right| 1 \right) \]
for \(0 \leq i, j \leq d\). These \(u_i\) are the dual Hahn polynomials.

Example 35.12 (Krawtchouk) Assume
\[ \theta_i = \theta_0 + si, \]
\[ \theta_i^* = \theta_0^* + s^*i \]
for \(0 \leq i \leq d\) and
\[ \varphi_i = ri(i-d-1) \]
\[ \phi_i = (r - ss^*)i(i-d-1) \]
for $1 \leq i \leq d$. Assume $r, s, s^*$ are nonzero. Assume the characteristic of $\mathbb{K}$ is 0 or a prime greater than $d$. Assume $r \neq ss^*$. Then the sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. The corresponding polynomials $u_i$ satisfy

$$u_i(\theta_j) = 2F_1\left(\begin{array}{c}
-i, -j \\
-i - d
\end{array} \left| r^{-1}ss^* \right. \right).$$

for $0 \leq i, j \leq d$. These $u_i$ are the Krawtchouk polynomials.

Example 35.13 (Bannai/Ito) Assume

\begin{align}
\theta_i & = \theta_0 + h(s - 1 + (1 - s + 2i)(-1)^i), \\
\theta_i^* & = \theta_0^* + h^*(s^* - 1 + (1 - s^* + 2i)(-1)^i)
\end{align}

for $0 \leq i \leq d$ and

\begin{align}
\varphi_i & = \begin{cases}
-4hh^*i(i + r_1), & \text{if } i \text{ even, } d \text{ even}; \\
-4hh^*(i - d - 1)(i + r_2), & \text{if } i \text{ odd, } d \text{ even}; \\
-4hh^*i(i - d - 1), & \text{if } i \text{ even, } d \text{ odd}; \\
-4hh^*(i + r_1)(i + r_2), & \text{if } i \text{ odd, } d \text{ odd},
\end{cases} \\
\phi_i & = \begin{cases}
4hh^*(i - s - s^* - r_1), & \text{if } i \text{ even, } d \text{ even}; \\
4hh^*(i - d - 1)(i - s^* - r_2), & \text{if } i \text{ odd, } d \text{ even}; \\
-4hh^*(i - d - 1), & \text{if } i \text{ even, } d \text{ odd}; \\
-4hh^*(i - s - s^* - r_1)(i - s^* - r_2), & \text{if } i \text{ odd, } d \text{ odd}
\end{cases}
\end{align}

for $1 \leq i \leq d$. Assume $h, h^*$ are nonzero and that $r_1 + r_2 = -s - s^* + d + 1$. Assume the characteristic of $\mathbb{K}$ is either 0 or an odd prime greater than $d/2$. Assume neither of $r_1, -s^* - r_1$ is equal to $-i$ for $1 \leq i \leq d$, $d - i$ even. Assume neither of $r_2, -s^* - r_2$ is equal to $-i$ for $1 \leq i \leq d$, $d - i$ odd. Assume neither of $s, s^*$ is equal to $2i$ for $1 \leq i \leq d$. Then the sequence $(\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d)$ is a parameter array over $\mathbb{K}$. We call the corresponding polynomials from Definition 35.1 the Bannai/Ito polynomials [11, p. 260].

Example 35.14 (Orphan) For this example assume $\mathbb{K}$ has characteristic 2. For notational convenience we define some scalars $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ in $\mathbb{K}$. We define $\gamma_i = 0$ for $i \in \{0, 3\}$ and $\gamma_i = 1$ for $i \in \{1, 2\}$. Assume

\begin{align}
\theta_i & = \theta_0 + h(si + \gamma_i), \\
\theta_i^* & = \theta_0^* + h^*(s^*i + \gamma_i)
\end{align}

for $0 \leq i \leq 3$. Assume $\varphi_1 = hh^*r$, $\varphi_2 = hh^*$, $\varphi_3 = hh^*(r+ s + s^*)$ and $\phi_1 = hh^*(r+ s^*(1 + s))$, $\phi_2 = hh^*$, $\phi_3 = hh^*(r + s^*(1 + s))$. Assume each of $h, h^*, s, s^*, r$ is nonzero. Assume neither of $s, s^*$ is equal to 1 and that $r$ is equal to none of $s + s^*$, $s(1 + s^*)$, $s^*(1 + s)$. Then the sequence $(\theta_i, \theta_i^*, i = 0..3; \varphi_j, \phi_j, j = 1..3)$ is a parameter array over $\mathbb{K}$ which has diameter 3. We call the corresponding polynomials from Definition 35.1 the orphan polynomials.

Theorem 35.15 Every parameter array over $\mathbb{K}$ is listed in at least one of the Examples 35.9–35.14.
Proof: Let \( p := (\theta_i, \theta_i^*, i = 0..d; \varphi_j, \phi_j, j = 1..d) \) denote a parameter array over \( \mathbb{K} \). We show this array is given in at least one of the Examples 35.2–35.14. We assume \( d \geq 1 \); otherwise the result is trivial. For notational convenience let \( \bar{\mathbb{K}} \) denote the algebraic closure of \( \mathbb{K} \). We define a scalar \( q \in \bar{\mathbb{K}} \) as follows. For \( d \geq 3 \), we let \( q \) denote a nonzero scalar in \( \bar{\mathbb{K}} \) such that

\[
q + q^{-1} + 1 = 0.
\]

For \( d < 3 \) we let \( q \) denote a nonzero scalar in \( \bar{\mathbb{K}} \) such that \( q \neq 1 \) and \( q \neq -1 \). By PA5, both

\[
\theta_{i-2} - \xi \theta_{i-1} + \xi \theta_i - \theta_{i+1} = 0, \tag{149}
\]
\[
\theta_{i-2}^* - \xi \theta_{i-1}^* + \xi \theta_i^* - \theta_{i+1}^* = 0. \tag{150}
\]

for \( 2 \leq i \leq d - 1 \), where \( \xi = q + q^{-1} + 1 \). We divide the argument into the following four cases. (I) \( q \neq 1, q \neq -1 \); (II) \( q = 1 \) and \( \text{char}(\mathbb{K}) \neq 2 \); (III) \( q = -1 \) and \( \text{char}(\mathbb{K}) \neq 2 \); (IV) \( q = 1 \) and \( \text{char}(\mathbb{K}) = 2 \).

Case I: \( q \neq 1, q \neq -1 \).

By (149) there exist scalars \( \eta, \mu, h \) in \( \bar{\mathbb{K}} \) such that

\[
\theta_i = \eta + \mu q^i + h q^{-i} \quad (0 \leq i \leq d). \tag{151}
\]

By (150) there exist scalars \( \eta^*, \mu^*, h^* \) in \( \bar{\mathbb{K}} \) such that

\[
\theta_i^* = \eta^* + \mu^* q^i + h^* q^{-i} \quad (0 \leq i \leq d). \tag{152}
\]

Observe \( \mu, h \) are not both 0; otherwise \( \theta_1 = \theta_0 \) by (151). Similarly \( \mu^*, h^* \) are not both 0. For \( 1 \leq i \leq d \) we have \( q^i \neq 1 \); otherwise \( \theta_i = \theta_0 \) by (151). Setting \( i = 0 \) in (151), (152) we obtain

\[
\theta_0 = \eta + \mu + h, \tag{153}
\]
\[
\theta_0^* = \eta^* + \mu^* + h^*. \tag{154}
\]

We claim there exists \( \tau \in \bar{\mathbb{K}} \) such that both

\[
\varphi_i = (q^i - 1)(q^{d-i+1} - 1)(\tau - \mu \mu^* q^{i-1} - hh^* q^{-i-d}), \tag{155}
\]
\[
\phi_i = (q^i - 1)(q^{d-i+1} - 1)(\tau - h \mu^* q^{i-1} - \mu h^* q^{-i}) \tag{156}
\]

for \( 1 \leq i \leq d \). Since \( q \neq 1 \) and \( q^d \neq 1 \) there exists \( \tau \in \bar{\mathbb{K}} \) such that (155) holds for \( i = 1 \). In the equation of PA4, we eliminate \( \varphi_1 \) using (155) at \( i = 1 \), and evaluate the result using (151), (152) in order to obtain (156) for \( 1 \leq i \leq d \). In the equation of PA3, we eliminate \( \phi_1 \) using (156) at \( i = 1 \), and evaluate the result using (151), (152) in order to obtain (155) for \( 1 \leq i \leq d \). We have now proved the claim. We now break the argument into subcases. For each subcase our argument is similar. We will discuss the first subcase in detail in order to give the idea; for the remaining subcases we give the essentials only.

Subcase q-Racah: \( \mu \neq 0, \mu^* \neq 0, h \neq 0, h^* \neq 0 \). We show \( p \) is listed in Example 35.2. Define

\[
s := \mu h^{-1} q^{-1}, \quad s^* := \mu^* h^{* -1} q^{-1} \tag{157}
\]

Eliminating \( \eta \) in (151) using (153) and eliminating \( \mu \) in the result using the equation on the left in (157), we obtain (135) for \( 0 \leq i \leq d \). Similarly we obtain (136) for \( 0 \leq i \leq d \). Since \( \bar{\mathbb{K}} \) is algebraically closed it contains scalars \( r_1, r_2 \) such that both

\[
r_1 r_2 = ss^* q^{d+1}, \quad r_1 + r_2 = \tau h^{-1} h^{* -1} q^d. \tag{158}
\]
Eliminating $\mu, \mu^*, \tau$ in (155), (156) using (157) and the equation on the right in (158), and evaluating the result using the equation on the left in (158), we obtain (137), (138) for $1 \leq i \leq d$. By the construction each of $h_i^*, h_i, q, s, s^*$ is nonzero. Each of $r_1, r_2$ is nonzero by the equation on the left in (158). The remaining inequalities mentioned below (138) follow from PA1, PA2 and (135)–(138). We have now shown $p$ is listed in Example 35.2.

We now give the remaining subcases of Case I. We list the essentials only.

Subcase $q$-Hahn: $\mu = 0, \mu^* \neq 0, h \neq 0, h^* \neq 0, \tau \neq 0$. Definitions:

$$s^* := \mu^* h^* q^{-1}, \quad r := \tau h^{-1} h^* q^d.$$

Subcase dual $q$-Hahn: $\mu \neq 0, \mu^* = 0, h \neq 0, h^* \neq 0, \tau \neq 0$. Definitions:

$$s := \mu h^{-1} q^{-1}, \quad r := \tau h^{-1} h^* q^d.$$

Subcase quantum $q$-Krawtchouk: $\mu \neq 0, \mu^* = 0, h = 0, h^* \neq 0, \tau \neq 0$. Definitions:

$$s := \mu q^{-1}, \quad r := \tau h^* q^d.$$

Subcase $q$-Krawtchouk: $\mu = 0, \mu^* \neq 0, h \neq 0, h^* \neq 0, \tau = 0$. Definition:

$$s^* := \mu^* h^* q^{-1}.$$

Subcase affine $q$-Krawtchouk: $\mu = 0, \mu^* = 0, h \neq 0, h^* \neq 0, \tau \neq 0$. Definition:

$$r := \tau h^{-1} h^* q^d.$$

Subcase dual $q$-Krawtchouk: $\mu \neq 0, \mu^* = 0, h \neq 0, h^* \neq 0, \tau = 0$. Definition:

$$s := \mu h^{-1} q^{-1}.$$

We have a few more comments concerning Case I. Earlier we mentioned that $\mu, h$ are not both 0 and that $\mu^*, h^*$ are not both 0. Suppose one of $\mu, h$ is 0 and one of $\mu^*, h^*$ is 0. Then $\tau \neq 0$; otherwise $\varphi_1 = 0$ by (155) or $\phi_1 = 0$ by (156). Suppose $\mu^* \neq 0, h^* = 0$. Replacing $q$ by $q^{-1}$ we obtain $\mu^* = 0, h^* \neq 0$. Suppose $\mu^* \neq 0, h^* = 0, \mu \neq 0, h = 0$. Replacing $q$ by $q^{-1}$ we obtain $\mu^* \neq 0, h^* \neq 0, \mu = 0, h \neq 0$. By these comments we find that after replacing $q$ by $q^{-1}$ if necessary, one of the above subcases holds. This completes our argument for Case I.

Case II: $q = 1$ and $\text{char}(\mathbb{K}) \neq 2$.

By (149) and since $\text{char}(\mathbb{K}) \neq 2$, there exist scalars $\eta, \mu, h$ in $\bar{\mathbb{K}}$ such that

$$\theta_i = \eta + (\mu + h)i + hi^2 \quad (0 \leq i \leq d). \quad (159)$$

Similarly there exist scalars $\eta^*, \mu^*, h^*$ in $\bar{\mathbb{K}}$ such that

$$\theta_i^* = \eta^* + (\mu^* + h^*)i + h^*i^2 \quad (0 \leq i \leq d). \quad (160)$$

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Observe $\mu, h$ are not both 0; otherwise $\theta_1 = \theta_0$. Similarly $\mu^*, h^*$ are not both 0. For any prime $i$ such that $i \leq d$ we have $\text{char}(\mathbb{K}) \neq i$; otherwise $\theta_i = \theta_0$ by (159). Therefore $\text{char}(\mathbb{K})$ is 0 or a prime greater than $d$. Setting $i = 0$ in (159), (160) we obtain

$$\theta_0 = \eta, \quad \theta_0^* = \eta^*.$$  \hspace{1cm} (161)

We claim there exists $\tau \in \bar{\mathbb{K}}$ such that both

$$\varphi_i = i(d - i + 1)(\tau - (\mu h^* + h\mu^*)i - hh^*i(i + d + 1)), \hspace{1cm} (162)$$

$$\phi_i = i(d - i + 1)(\tau + \mu h^* + h\mu^*(1 + d) + (\mu h^* - h\mu^*)i + hh^*i(d - i + 1)). \hspace{1cm} (163)$$

for $1 \leq i \leq d$. There exists $\tau \in \bar{\mathbb{K}}$ such that (162) holds for $i = 1$. In the equation of PA4, we eliminate $\varphi_1$ using (162) at $i = 1$, and evaluate the result using (159), (160) in order to obtain (163) for $1 \leq i \leq d$. In the equation of PA3, we eliminate $\phi_1$ using (163) at $i = 1$, and evaluate the result using (159), (160) in order to obtain (162) for $1 \leq i \leq d$. We have now proved the claim. We now break the argument into subcases.

Subcase Racah: $h \neq 0, h^* \neq 0$. We show $p$ is listed in Example 35.9. Define

$$s := \mu h^{-1}, \quad s^* := \mu^* h^*^{-1}. \hspace{1cm} (164)$$

Eliminating $\eta, \mu$ in (159) using (161), (164) we obtain (139) for $0 \leq i \leq d$. Eliminating $\eta^*, \mu^*$ in (160) using (161), (164) we obtain (140) for $0 \leq i \leq d$. Since $\bar{\mathbb{K}}$ is algebraically closed it contains scalars $r_1, r_2$ such that both

$$r_1 r_2 = -\tau h^{-1} h^*^{-1}, \quad r_1 + r_2 = s + s^* + d + 1. \hspace{1cm} (165)$$

Eliminating $\mu, \mu^*, \tau$ in (162), (163) using (164) and the equation on the left in (165) we obtain (141), (142) for $1 \leq i \leq d$. By the construction each of $h, h^*$ is nonzero. The remaining inequalities mentioned below (142) follow from PA1, PA2 and (139)–(142). We have now shown $p$ is listed in Example 35.9.

We now give the remaining subcases of Case II. We list the essentials only.

Subcase Hahn: $h = 0, h^* \neq 0$. Definitions:

$$s = \mu, \quad s^* := \mu^* h^*^{-1}, \quad r := -\tau h^{-1} h^*^{-1}. \hspace{1cm} (166)$$

Subcase dual Hahn: $h \neq 0, h^* = 0$. Definitions:

$$s := \mu h^{-1}, \quad s^* = \mu^*, \quad r := -\tau h^{-1} \mu^*.$$  

Subcase Krawtchouk: $h = 0, h^* = 0$. Definitions:

$$s := \mu, \quad s^* := \mu^*, \quad r := -\tau.$$

Case III: $q = -1$ and $\text{char}(\mathbb{K}) \neq 2$.

We show $p$ is listed in Example 35.13. By (149) and since $\text{char}(\mathbb{K}) \neq 2$, there exist scalars $\eta, \mu, h$ in $\bar{\mathbb{K}}$ such that

$$\theta_i = \eta + \mu(-1)^i + 2hi(-1)^i \quad (0 \leq i \leq d). \hspace{1cm} (166)$$

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Similarly there exist scalars $\eta^*, \mu^*, h^*$ in $\overline{\mathbb{K}}$ such that
\[
\theta_i^* = \eta^* + \mu^*(-1)^i + 2h^*i(-1)^i \quad (0 \leq i \leq d).
\tag{167}
\]

Observe $h \neq 0$; otherwise $\theta_2 = \theta_0$ by (166). Similarly $h^* \neq 0$. For any prime $i$ such that $i \leq d/2$ we have $\text{char}(\mathbb{K}) \neq i$; otherwise $\theta_{2i} = \theta_0$ by (166). By this and since $\text{char}(\mathbb{K}) \neq 2$ we find $\text{char}(\overline{\mathbb{K}})$ is either 0 or an odd prime greater than $d/2$. Setting $i = 0$ in (166), (167) we obtain
\[
\theta_0 = \eta + \mu, \quad \theta_0^* = \eta^* + \mu^*.
\tag{168}
\]

We define
\[
s := 1 - \mu h^{-1}, \quad s^* = 1 - \mu^* h^{-1}.
\tag{169}
\]

Eliminating $\eta$ in (166) using (168) and eliminating $\mu$ in the result using (169) we find (143) holds for $0 \leq i \leq d$. Similarly we find (144) holds for $0 \leq i \leq d$. We now define $r_1, r_2$. First assume $d$ is odd. Since $\overline{\mathbb{K}}$ is algebraically closed it contains $r_1, r_2$ such that
\[
r_1 + r_2 = -s - s^* + d + 1
\tag{170}
\]
and such that
\[
4hh^*(1 + r_1)(1 + r_2) = -\varphi_1.
\tag{171}
\]

Next assume $d$ is even. Define
\[
r_2 := -1 + \frac{\varphi_1}{4hh^*d}
\tag{172}
\]
and define $r_1$ so that (170) holds. We have now defined $r_1, r_2$ for either parity of $d$. In the equation of PA4, we eliminate $\varphi_1$ using (171) or (172), and evaluate the result using (143), (144) in order to obtain (146) for $1 \leq i \leq d$. In the equation of PA3, we eliminate $\phi_1$ using (146) at $i = 1$, and evaluate the result using (143), (144) in order to obtain (145) for $1 \leq i \leq d$. We mentioned each of $h, h^*$ is nonzero. The remaining inequalities mentioned below (146) follow from PA1, PA2 and (143)–(146). We have now shown $p$ is listed in Example 35.13.

Case IV: $q = 1$ and $\text{char}(\mathbb{K}) = 2$.

We show $p$ is listed in Example 35.14. We first show $d = 3$. Recall $d \geq 3$ since $q = 1$. Suppose $d \geq 4$. By (149) we have $\sum_{j=0}^3 \theta_j = 0$ and $\sum_{j=1}^4 \theta_j = 0$. Adding these sums we find $\theta_0 = \theta_4$ which contradicts PA1. Therefore $d = 3$. We claim there exist nonzero scalars $h, s$ in $\mathbb{K}$ such that (147) holds for $0 \leq i \leq 3$. Define $h = \theta_0 + \theta_2$. Observe $h \neq 0$; otherwise $\theta_0 = \theta_2$. Define $s = (\theta_0 + \theta_3)h^{-1}$. Observe $s \neq 0$; otherwise $\theta_0 = \theta_3$. Using these values for $h, s$ we find (147) holds for $i = 0, 2, 3$. By this and $\sum_{j=0}^3 \theta_j = 0$ we find (147) holds for $i = 1$. We have now proved our claim. Similarly there exist nonzero scalars $h^*, s^*$ in $\mathbb{K}$ such that (148) holds for $0 \leq i \leq 3$. Define $r := \varphi_1 h^{-1}h^*$, $\phi_1 = hh^*r$ in $\mathbb{K}$, and evaluate the result using (147), (148) in order to obtain $\phi_1 = hh^*(r + s(1 + s^*))$, $\phi_2 = hh^*$, $\phi_3 = hh^*(r + s^*(1 + s))$. In the equation of
PA3, we eliminate $\phi_1$ using $\phi_1 = hh^*(r+s(1+s^*))$ and evaluate the result using (147), (148) in order to obtain $\varphi_2 = hh^*$, $\varphi_3 = hh^*(r+s+s^*)$. We mentioned each of $h, h^*, s, s^*, r$ is nonzero. Observe $s \neq 1$; otherwise $\theta_1 = \theta_0$. Similarly $s^* \neq 1$. Observe $r \neq s + s^*$; otherwise $\varphi_3 = 0$. Observe $r \neq s(1+s^*)$; otherwise $\phi_1 = 0$. Observe $r \neq s^*(1+s)$; otherwise $\phi_3 = 0$. We have now shown $p$ is listed in Example 35.14. We are done with Case IV and the proof is complete.

### 36 Suggestions for further research

In this section we give some suggestions for further research.

**Problem 36.1** Let $V$ denote a vector space over $\mathbb{K}$ with finite positive dimension and let $A, A^*$ denote a tridiagonal pair on $V$. Let $\alpha, \alpha^*, \beta, \beta^*$ denote scalars in $\mathbb{K}$ with $\alpha, \alpha^*$ nonzero, and note that the pair $\alpha A + \beta I, \alpha^* A^* + \beta^* I$ is a tridiagonal pair on $V$. Find necessary and sufficient conditions for this tridiagonal pair to be isomorphic to the tridiagonal pair $A, A^*$. Also, find necessary and sufficient conditions for this tridiagonal pair to be isomorphic to the tridiagonal pair $A^*, A$. This problem has been solved for Leonard pairs [81].

**Problem 36.2** Assume $\mathbb{K} = \mathbb{R}$. With reference to Definition 15.1, find a necessary and sufficient condition on the parameter array of $\Phi$, for the bilinear form $\langle \cdot, \cdot \rangle$ to be positive definite. By definition the form $\langle \cdot, \cdot \rangle$ is positive definite whenever $\|u\|^2 > 0$ for all nonzero $u \in V$.

In order to motivate the next problem we make a definition.

**Definition 36.3** Let $\Phi$ denote the Leonard system from Definition 3.2. For $0 \leq i \leq d$ we define $A_i = v_i(A)$, where the polynomial $v_i$ is from Definition 13.1. Observe that there exist scalars $p_{ij}^h \in \mathbb{K}$ ($0 \leq h, i, j \leq d$) such that

$$A_i A_j = \sum_{h=0}^d p_{ij}^h A_h \quad (0 \leq i, j \leq d).$$

We call the $p_{ij}^h$ the *intersection numbers* of $\Phi$.

**Problem 36.4** Let $\Phi$ denote the Leonard system from Definition 3.2. For each of the Examples 35.2–35.14, if possible express each intersection number as a hypergeometric series or a basic hypergeometric series. Also for $\mathbb{K} = \mathbb{R}$, determine those $\Phi$ for which the intersection numbers are all nonnegative.

**Problem 36.5** Assume $\mathbb{K} = \mathbb{R}$ and let $\Phi$ denote the Leonard system from Definition 3.2. Determine those $\Phi$ for which the intersection numbers of each of $\Phi, \Phi^1, \Phi^2, \Phi^3$ are all nonnegative. Also, determine those $\Phi$ for which the intersection numbers of each relative of $\Phi$ are all nonnegative.

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Next goal. Find the $b_i$, $c_i$ in terms of $A$.

Given $LS$,

$$L = (A, E, A^*, E_i)$$

with $PA$

$$p_i \cdot \psi_i \quad \psi_i \quad \phi_i$$

Recall $c_i, b_i \in F$ satisfy

$$x u_i = c_i \cdot u_i + a_i \cdot u_i + b_i \cdot u_i  \quad 0 \leq i \leq N - 1$$

Also

$$x u_N - c_N \cdot u_N - a_N \cdot u_N \in F \text{ with } u_N \equiv 1 \text{ mod } |A|$$

**LEM 2.4.6**

$$b_i = c_i = \frac{T_i^*(\phi_i)}{T_i^*(\phi_i^*)} \quad 0 \leq i \leq N - 1$$

1. $$c_i = \phi_i \quad x \quad \frac{\gamma_{N-i}^*(\phi_i^*)}{\gamma_{N-i}^*(\phi_i^*)} \quad 1 \leq i \leq N$$

**pf (i)** We have seen

$$b_{0b_0} = b_0 = \psi_i(\phi_0)$$

$$= \psi_i \psi_2 \psi_i \quad \psi_i \quad T_i^*(\phi_i)$$

Result follows by induction on $i$.

**cii** Apply (i) to $\overline{\psi}^*$ and note

$$c_i(\overline{\psi}^*) = b_{N-i}(\overline{\psi})$$
Another formula for $b_i, c_i$

(Assume $N \geq 1$ to avoid divisions)

LEM 247 \quad \text{For } N \geq 1

\[ c_i \left( \theta_{iT}^* - \theta_0^* \right) - b_i \left( \theta_i^* - \theta_{iT}^* \right) = (q - q_0) \left( \theta_i^* - \theta_0^* \right) + \psi \]

\[ \sigma \leq i \leq N \]

(\theta_i^*, \theta_{iN}^* \text{ undefined})

pf \quad \text{In th. 238 set } \kappa = \theta_i \text{ to get}

\[ u_{\theta_i}(\theta_i) = 1 + \frac{(\theta_i - \theta_0) (\theta_i^* - \theta_0^*)}{\psi} \]

\[ 0 \leq i \leq N \]

In the 3-term rec

\[ \theta_i u_{\theta_i}(\theta_i) = c_i u_{\theta_i}(\theta_i) + a_i u_{\theta_i}(\theta_i) + \psi \text{u} \text{m} \text{r} \text{l}_1 \]

\[ 0 \leq i \leq N \]

elim $\theta_i(\theta_i)$ using * and $a_i$ using

\[ a_i = \theta_0 - b_i - c_i \]
\[ b_0 = \frac{\psi_i}{\theta_i^*-\theta_o^*} \]

\[ b_i' = \frac{\left( \theta_0 - \alpha_i \right) \left( \theta_i^*-\theta_i^* \right) + \left( \theta_0 - \alpha_i \right) \left( \theta_i^*-\theta_i^* \right)}{\theta_i^*-\theta_i^*} \]

\[ 1 \leq i \leq N-1 \]

\[ c_i' = \frac{\left( \theta_0 - \alpha_i \right) \left( \theta_i^*-\theta_i^* \right) + \left( \theta_0 - \alpha_i \right) \left( \theta_i^*-\theta_i^* \right)}{\theta_i^*-\theta_i^*} \]

\[ 1 \leq i \leq N-1 \]

\[ c_N = \frac{\phi_N}{\theta_N^*-\theta_N^*} \]

[Remark: \( \alpha_i \) given in L246]

pf (i) \[ \text{Set } i = 0 \text{ in } L246 \]

\[ \text{Solve the system} \]

\[ \begin{cases} c_0 + b_0 = \theta_0 - \alpha_i \\ \text{L246} \end{cases} \]

for \( c_0, b_0 \)

\[ \text{Set } i = N \text{ in } L246 \]

\[ \square \]
Ex 249 Assume $\mathbb{E}$ has Krauthkopf type \( c \)

\[ \theta_1 = c \quad \theta_2 = c \quad \sigma \in \mathbb{N} \]

\[ \text{pm} \]

(i) \[ b_1 = (x' - N)p \quad 0 \leq x' \leq N \]

(ii) \[ c_{x_1} = c'(p - r) \quad 0 \leq i \leq N \]

(iii) \[ a_{x_1} = c'(1 - p) + (N - c')p \quad 0 \leq i \leq N \]

Same data as from p. 99

pf PA given in L244

Use either L246 and

\[ c_{x_1} + b_1 = a_0 \quad 0 \leq i \leq N \]

or

L248 and L2410

\[ \square \]
Next goal:

For the terminating branch of Askey scheme,

a uniform treatment of orthogonality using corner LS.

Until further notice for LS

\[ \mathcal{F} = (A, \{ E^\alpha_{i=0} \}, A^a, \{ E^\alpha_{i=0} \} ) \quad \text{on} \quad V \]

For array

\[
(\{ e_i \}_{i=0}^N, \{ a_{i=0}^N \}, \{ b_{i=0}^N \}, \{ d_{i=0}^N \})
\]

Def 250 Def

\[ m_\alpha = \text{tr}(E, E_\alpha^b) \quad 0 \leq \alpha \leq N \]
LEM 251

(i) \[ E_i E_0^r E_i' = m_r E_i \quad 0 \leq i \leq N \]

(ii) \[ E_0^r E_i E_0^r = m_r E_0^r \quad 0 \leq i \leq N \]

(iii) \[ m_r \neq 0 \quad 0 \leq r \leq N \]

(iv) \[ \sum_{i=0}^{N} m_i = 1 \]

(v) \[ m_0 = m_0^r \]

pf. (i) Since \( E_i \) is rank 1 idempotent

\[ E_i A E_i = A \quad A = \text{End} V \]

So \( \exists \alpha_i \in \mathbb{F} \) s.t.

\[ E_i E_0^r E_i' = \alpha_i E_i \]

In this eq take trace to get \( \alpha_i = m_i \)

(iii) \[ E_0^r \] is normalising so \( E_0^r E_0^r \neq 0 \)

Apply \( \text{Tr} \) to get

\[ E_0^r E_0^r \neq 0 \]

So \( E_0^r E_i V = E_i V \)

Now

\[ E_i E_0^r E_i' \neq 0 \]

since

\[ E_i E_0^r E_i V = E_i E_0^r V \neq 0 \]

Now \( m_r \neq 0 \) by (i)

(iv) \[ \sum_{i=0}^{N} m_i = \text{Tr} \sum_{i=0}^{N} E_i E_0^r = \text{Tr} E_0^r = 1 \]

\[ \quad \text{case trace.} \]
Def 252  Put
\[ \mathcal{Y} = \frac{1}{m_0} = \frac{1}{m_0^*} \]

So
\[ \mathcal{Y}^* = 1 \cdot (E_0 E_0^*) \]

Obs
\[ \mathcal{Y} = \mathcal{Y}_* \]

Lemma 253  We have
\[ (i) \quad \mathcal{Y} E_0^* E_0 E_0^* = E_0^* \]
\[ (ii) \quad \mathcal{Y} E_0 E_0^* E_0 = E_0 \]

pf
Set \( \tau = 0 \) in Lem 251 (i), (ii)
\[ \square \]
\[ V = \frac{\gamma^\ast_N (\theta_0) \gamma^\ast_N (\theta_0^\ast)}{\phi_1 \phi_2 \cdots \phi_N} \]

\[ = \frac{(\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_N)(\theta_0^\ast - \theta_1^\ast)(\theta_0^\ast - \theta_2^\ast) \cdots (\theta_0^\ast - \theta_N^\ast)}{\phi_1 \phi_2 \cdots \phi_N} \]

\[ \text{pf} \]

Pick \( e + v \in E_0 \times V \).

Recall \( \{ \gamma_i : A_i V \}_{i=0}^N \) is basis for \( V \). Rel this basis.

\[ A : \begin{pmatrix} \theta_0^\ast & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \theta_0 \end{pmatrix} \]

\[ A^\ast : \begin{pmatrix} \theta_0^\ast & \phi_1^\ast & \cdots & \phi_N^\ast \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ \phi_N \end{pmatrix} \]

Rep \( E_0 : E_0^\times \) the basis.

Form is

\[ E_0 \times \begin{pmatrix} \theta_0^\ast & \cdots & \phi_N^\ast \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ \phi_N \end{pmatrix} \]

\[ E_0 E_0^\times : \begin{pmatrix} \theta_0^\ast & \cdots & \phi_N^\ast \end{pmatrix} \begin{pmatrix} 1 & \cdots & \beta \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \]

\[ V^\ast = \mathbb{E}_0 E_0^\times = \kappa \beta \]

\[ A_{20} = \begin{pmatrix} 20 & \ast & \ast & \ast \end{pmatrix} \begin{pmatrix} 1 \cdots \beta \\ 1 \cdots \beta \end{pmatrix} \]

\[ \gamma_0 (\theta_0) \]

\[ \beta = \phi_1 \phi_2 \cdots \phi_N \]
Ex 255 Assume $F$ has Krawtchouk type $s$

$$\theta_s = c, \quad \theta_s^* = c', \quad 0 \leq s \leq n$$

Then

$$v = (1 - p)^{-n}$$

where $p$ is from L244

pf Eval th 254 using L244
Recall
\[ k_i = \frac{b_{0i}^i \ldots \ b_{ni}^i}{c_1 \ c_2 \ldots \ c_i} \quad o e i e N \]

**Lem 256** \( \Gamma_n \quad o e i e N \)

\[ k_i = \nabla \alpha (E_i^x E_i) \]

\( = \nabla m_i^x \)

**pf**

Pick \( o u e E_0V \)

Recall \( E - \) stand basis \( \Gamma V \)

\[ E_i^x u \quad o e i e N \]

Recall \( \{ v_i \}_{i=0}^N \) sat

\[ v_i(A) E_0^x u = E_i^x u \quad o e i e N \]

**Obs**

\[ m_i^x u = m_i^x E_0^x u \]

\[ = E_0 E_i^x E_0 u \]

\[ = E_0 E_i^x u \]

\[ = E_0 v_i(A) E_0^x u \]

\[ = v_i(\theta_0) E_0 E_0^x u \]

\[ = v_i(\theta_0) \frac{E_0 E_0^x E_0 u}{\nabla E_0} \]

\[ \begin{bmatrix} v_i = k_i u \theta_0 \quad u_0(\theta_0) = 1 \\ v_i(\theta_0) = k_i \end{bmatrix} \]

\[ = k_i \nabla E_0 u \]

\[ = k_i \nabla u \quad s \quad m_i^x = \lambda_i \nabla \]
We define a bilinear form

\[ \langle \cdot, \cdot \rangle : V \times V \to \mathbb{F} \]

as follows.

Pick \( o \neq u \in \mathbb{F} \cdot V \).

Recall the standard basis for \( V \):

\[ E_i^x U \quad 0 \leq i \leq n \]

We define \( \langle \cdot, \cdot \rangle \) on \( \mathbb{F} \).

Put \( o \neq f \in \mathbb{F} \).

Put

\[ \langle E_i^x u, E_j^x u \rangle = \delta_{ij} k_i f \quad 0 \leq i, j \leq n \]

so matrix \( \langle \cdot, \cdot \rangle \) is

\[ K = f \text{ diag } (k_i) \]

Observe

\[ \langle \cdot, \cdot \rangle \text{ is sym} \]

\[ \langle \cdot, \cdot \rangle \text{ is non deg since } K^{-1} \text{ exists} \]

Observe \( k_0 = 1 \) so

\[ \|E_0^x u\|_2^2 = f \]

so

\[ \langle E_i^x u, E_j^x u \rangle = \delta_{ij} k_i \|E_0^x u\|_2^2 \quad 0 \leq i, j \leq n \]
LEM 257 \( \forall x \in \text{End} V \quad \forall w \in V \)

\[
\langle Xv, w \rangle = \langle v, X^w \rangle
\]

where \( t \) is antiadjoint to \( \text{End} V \) from L188

Proof By L1187

\[
A, E_0^x \quad \text{gen} \quad \text{End} V
\]

WLOG \( X = A \quad \text{and} \quad X = E_0^x \)

Recall \( A^t = A \quad E_0^{x^t} = E_0^x \)

Put \( e^t \in E_0 V \) Consider \( \overline{A} \)-shift basis \( \{ \cdot \} \) \( V \)

\[
E_i^x \quad 0 \leq i \leq N
\]

WLOG \( v, w \in \text{O} \)

Recall metric \( \text{eg} A, E_0^x \text{eg} \)

\[
A \colon \begin{pmatrix}
a_0 & b_0 & \cdots & b_1 \\
c_0 & a_1 & \cdots & c_1 \\
\vdots & \vdots & \ddots & \vdots \\
c_n & c_{n-1} & \cdots & a_n
\end{pmatrix}
\]

\( E_0^x \colon \text{diag}(1,0,\ldots,0) \)

\[
\langle Xv, w \rangle = \langle v, X^w \rangle
\]

Case \( X = A \)

\[
\langle AE_i^x u, E_j^x u \rangle = \langle E_i^x u, AE_j^x u \rangle \quad 0 \leq i, j \leq N
\]

Holds since

\[
K^t B = B^t K
\]

Case \( X = E_0^x \) routine
LEM 258

(\text{i}) \quad \text{For } u \in E_0 \forall \quad \|E_0^* u\|^2 = \nu^{-2} \|u\|^2

(\text{iii}) \quad \text{For } v \in E_0^* V \quad \|E_0 v\|^2 = \nu^{-2} \|v\|^2

\text{pf (\text{i})} \quad \text{Observe } E_0 u = u

\|E_0^* u\|^2 = \langle E_0^* u, E_0^* u \rangle

= \langle E_0^* E_0 u, E_0^* E_0 u \rangle

= \langle u, \frac{E_0 E_0^* E_0 u}{\|v^{-1} E_0\|} \rangle

= \nu^{-2} \langle u, u \rangle

(\text{iii}) \quad \text{Similarly}

\square
Pick $0 \neq v \in E_0^tv$

Consider $E^* -$ stand basis of $V$:

$$E_i^tv \quad 0 \leq i \leq n$$

**LEM 259** With above notation

$$\langle E_i^tv, E_j^tv \rangle = \delta_{ij} k_i^x \parallel E_0v \parallel^2 \quad 0 \leq i \leq n$$

**pf** LHS = $$\langle E_i^x E_0^tv, E_j^x E_0^tv \rangle$$

= $$\langle v, \underbrace{E_0^x E_i E_j E_0^x v}_{\delta_{ij} m_i^x E_0^x} \rangle$$

= $$\delta_{ij} m_i^x \parallel v \parallel^2$$

= $$\delta_{ij} m_i^x \parallel E_0v \parallel^2$$

$$k_i^x \parallel E_0v \parallel^2 \quad \square$$
LEM 260 \( \text{ Let } \theta \in \mathbb{E}_0^V \text{ and } \theta \in \mathbb{E}_0^V \) 

\( (i) \) Each \( \theta \) \( \| u \|, \| v \| \), \( \langle u, v \rangle \) is real

\( (ii) \) \( E_0^u = \frac{\langle u, v \rangle v}{\| v \|^2} \)

\( (iii) \) \( E_0^v = \frac{\langle u, v \rangle u}{\| u \|^2} \)

\( (iv) \) \( \forall \langle u, v \rangle \), \( \| u \|^2 \| v \|^2 \)

pf \( (ii) \) \( v \) is basis \( \theta \) \( E_0^V \) so \( \exists \lambda \in \mathbb{F} \)

\( E_0^u = \lambda v \)

\( \langle E_0^u, u \rangle = \langle \lambda v, u \rangle \)

\( = \lambda \langle v, u \rangle \)

\( = \lambda \| v \|^2 \langle u, v \rangle \)

\( \lambda = \frac{\langle u, v \rangle^2}{\| v \|^2} \)

\( (iii) \) \( \text{ Similar} \)

\( (i) \) Since \( E_0^u \neq 0 \)

\( (iv) \) \( v \) \( \langle u, v \rangle = \langle u, E_0^x E_0^x E_0^x v \rangle \)

\( = \langle E_0^x u, E_0^x v \rangle \)

\( = \frac{\langle u, v \rangle^2}{\| u \|^2 \| v \|^2} \langle v, u \rangle \frac{\langle v, u \rangle}{\langle u, v \rangle} \)

\( \square \)
Thm 2.6.1 \( \forall u \in E_0^*V \text{ and } \forall v \in E_0^*V \)

\[
\left\langle E_i^*u, E_j^*v \right\rangle = \mathcal{V}^* k_i^* u_i(\theta_i) \left\langle u, v \right\rangle \quad 0 \leq i, j \leq N
\]

\[
\text{pf} \quad \left\langle E_i^*u, E_j^*v \right\rangle = \left\langle v_i(A) E_0^*u, E_j^*v \right\rangle
\]

\[
= \left\langle E_0^*u, v_i(A) E_j^*v \right\rangle
\]

\[
= v_i(\theta_i) \left\langle E_0^*u, E_j^*v \right\rangle
\]

\[
= v_i(\theta_i) \left\langle v_i(A) E_0^*u, E_0^*v \right\rangle
\]

\[
= v_i(\theta_i) \left\langle v_i(A) E_0^*u, E_0^*v \right\rangle
\]

\[
= k_j^* v_i(\theta_i) \left\langle E_0^*u, E_0^*v \right\rangle
\]

\[
= k_j^* u_i(\theta) \mathcal{V}^* u, v \right\rangle
\]
Theorem 2.6.2  
For $v + u \in E_0^v$ and $v + u \in E_0^u$,

(i) \[ E_i^v u = \frac{\langle u, v \rangle}{\| v \|^2} \sum_{j=0}^{N} \nu_i (\theta_j) E_j^v u \quad 0 \leq i \leq N \]

(ii) \[ E_i^u v = \frac{\langle u, v \rangle}{\| v \|^2} \sum_{j=0}^{N} \nu_i (\theta_j) E_j^u u \quad 0 \leq i \leq N \]

Proof (i) \[ E_i^v u = \nu_i (\Lambda) E_0^v u \]

\[ = \left( \sum_{j=0}^{N} E_j \right) \nu_i (\Lambda) E_0^v u \]

\[ = \sum_{j=0}^{N} \nu_i (\theta_j) E_j \left( \frac{E_0^v u}{\| u \|} \right) \frac{\langle u, v \rangle}{\| v \|^2} \]

Conclusion 5.1
We now give the orthogonality of the \( F_i \) for \( i = 0 \)

**Thm 263**

(i) \[ F_n \quad 0 \leq i, j \leq N \]

\[
\sum_{r=0}^{N} v_n^i(\vartheta_r) v_n^j(\vartheta_r) k_r^i = \delta_{ij} v_n^i k_r^i
\]

(ii) \[ F_n \quad 0 \leq i, j \leq N \]

\[
\sum_{s=0}^{N} v_s^i(\vartheta_s) v_i(\vartheta_s) k_s^j = \delta_{ij} v s k_r^j
\]

**pf** (i) Pick \( u \neq u \leq E_0 \quad u \neq u \leq E_0 \)

\[
\left< E_i^x u, E_j^x u \right> = \frac{\| u \|^2}{\| u \|^2} \sum_{r=0}^{N} v_r^i(\vartheta_r) v_r^j(\vartheta_r) \| E_r u \|^2
\]

**def:**

\[ \delta_{ij} k_r \| E_0^x u \|^2 \]

\[ \| E_0 u \|^2 \]

\[ \| u \|^2 \]

Simplify using \( L260 (iv) \)

(ii) Apply (i) to \( \mathbb{R}^x \) use Askey Wilsom duality
We now give the orthogonality for the $\{u_i\}_{i=0}^\infty$.

**Theorem 2.64**

(i) For $0 \leq i, j \leq N$,

$$
\sum_{r=0}^{\infty} u_i(r) u_j(r) k_r^x = \delta_{ij} \vee k_r^x
$$

(ii) For $0 \leq i, j \leq N$,

$$
\sum_{r=0}^{\infty} u_i(r) u_j(r) k_r^y = \delta_{ij} \vee k_r^y
$$

Proof: Eval $\text{Theorem 2.63}$ using

$$
V_h = U_h k_h \quad 0 \leq h \leq N
$$
Next goal: the difference equation is \( \sum_{i=0}^{N} x_i \).

(\textit{analog of 12.108})

To motivate, recall a 3-term rec:

\[
\theta_j u_i(\theta_j) = c_i u_{i+1}(\theta_j) + a_i u_{i}(\theta_j) + b_i u_{i-1}(\theta_j)
\]

Th 265 \( F_n \quad 0 \leq i \leq N \)

\[
\theta_i^x u_i(\theta_j) = c_i^x u_{i+1}(\theta_j) + a_i^x u_{i}(\theta_j) + b_i^x u_{i-1}(\theta_j)
\]

\( c_0^x = 0, \quad b_N^x = 0, \quad \theta_{\mathcal{F}}, \theta_{\mathcal{W}} \text{ undet} \)

pf. Apply \( \psi \) to \( \mathcal{F}^x \) and use Askey-Wilson duality. \( \square \)
Next goal: Express our results on Leonard systems / term branch of Askey scheme in terms of matrices.

**DEF 2.6** For our given $LS \Xi$, we define:

<table>
<thead>
<tr>
<th>Matrix Name</th>
<th>Matrix entries</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>$(U_{\Xi}(g))_{0 \leq i, j \leq N}$</td>
</tr>
</tbody>
</table>
| $B$         | \[
\begin{pmatrix}
    a_0 & b_0 & & & \\
    c_1 & a_1 & b_1 & & \\
    & c_2 & & & \\
    & & \ddots & & \\
    & & & c_N & a_N \\
    & & & & b_{N+1} \\
\end{pmatrix}
\] |
| $D$         | $\text{diag}(\theta_0, \theta_1, \ldots, \theta_N)$ |
| $K$         | $\text{diag}(k_0, k_1, \ldots, k_N)$ |

$k_i = \frac{b_0 + b_1 + \cdots + b_i}{c_0 + \cdots + c_i}$ for $0 \leq i \leq N$

$U^*, B^*, D^*, K^*$ are corresponding matrices.
<table>
<thead>
<tr>
<th>Result</th>
<th>Meaning</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U^* = U$</td>
<td>Askey-Wilson duality</td>
<td>Th 243</td>
</tr>
<tr>
<td>$B^* = KBK^*$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$UD = BU$</td>
<td>3-term rec</td>
<td>Above Thm 238</td>
</tr>
<tr>
<td>$D^* U = U B^* U$</td>
<td>difference equation</td>
<td>Thm 265</td>
</tr>
<tr>
<td>$v^* U K^* U^* K = I$</td>
<td>orthogonality</td>
<td>Thm 264 (a)</td>
</tr>
<tr>
<td>$v^* U^* K U K^* = I$</td>
<td></td>
<td>Thm 264 (c)</td>
</tr>
</tbody>
</table>

"Compare with 1116"
For the Krawtchouk polynomials we defined a matrix $P$ above.

We now define $P$ from (5).

DEF 268 \[ P \in \text{Mat}_{mn}(F) \] \; s.t.

\[ p_{ij} = v_{ij}(\theta) \quad 0 \leq i, j \leq n \]

Obs \[ P = U^t K = U^* K \]

\[ U = K^{-1} P^t = P^* K^{-1} \]
Theorem 2.69

With ref to Def 2.66 and Def 2.68

\[ p^t = K p^x k^{x^t} \]

\[ \beta^t = K B k^{x^t} \]

\[ p 0^x = B^x P \]

\[ p B = B P \]

\[ p p^x = \nu I \]

pf: In \( k 26 \) elim \( \mathcal{U} \) using Def 2.68 \( \square \)
Next goal: $\mathcal{F}$-dual standard basis

Def 270: Given $0 \neq u \in E_0 \setminus V$ observe

$$\frac{E_i^* u}{k_i}, \quad 0 \leq i \leq N$$

is basis for $V$. Call them $\phi_i$.

"$\mathcal{F}$-dual standard basis"

**LEM 271:** Fix $0 \neq u \in E_0 \setminus V$. Consider $\mathcal{F}$-standard basis

$$E_i^* u, \quad 0 \leq i \leq N$$

With respect to $\langle \cdot, \cdot \rangle$ the basis for $V$ dual $h_i$ is

$$\frac{u}{\|u\|^2}, \quad \frac{E_i^* u}{k_i}, \quad 0 \leq i \leq N$$

Moreover $\chi_2$ is a $\mathcal{F}$-dual standard basis

p+ Recall

$$\langle E_i^* u, E_j^* u \rangle = \delta_{ij} k_i \|E_i^* u\|^2, \quad 0 \leq i, j \leq N$$

and $\|E_0^* u\|^2 = \nu \|u\|^2$. 


LEM 272. Let \( \{ \omega_i \}_{i=0}^\infty \) denote a dual standard basis of \( V \).

Then the r.v. \( \{ \omega_i \}_{i=0}^\infty \) sat

\[
\omega_i (A) \omega_0 = \omega_i
\]

for \( i = 1, 2, \ldots \).

Set \( \omega_i = \frac{E_i^* u}{k_i} \) whenever \( i = 0, 1, \ldots \).

Recall

\[
E_i^* u = \omega_i (A) E_0 u
\]

\[
V_i = k_i \omega_i
\]

\[
k_0 = 1
\]
LEM 273. Let \( \{ w_i \}_{i=0}^N \) denote a \( F \)-dual stand basis for \( V \)
and \( \{ w_i^* \}_{i=0}^N \) denote \( F^* \)-dual stand basis.

\[
\langle w_i, w_j^* \rangle = u_i(\theta_0) \langle w_0, w_j^* \rangle \quad 0 \leq i, j \leq N
\]

**pf**

\( \exists \ o + v \in E_0 V \) s.t.

\[
w_i = \frac{E_i^* v}{k_i} \quad 0 \leq i \leq N
\]

\( \exists \ o + v \in E_0 V \) s.t.

\[
w_i^* = \frac{E_i v}{k_i^*} \quad 0 \leq i \leq N
\]

By A.261

\[
\langle E_i^* u, E_j v \rangle = v^* k_i k_j^* u_\ell(\theta_0) \langle u, v \rangle \quad 0 \leq i, j \leq N
\]

so

\[
\langle E_0^* u, E_0 v \rangle = v^* \langle u, v \rangle
\]

so

\[
\langle E_i^* u, E_0 v \rangle = k_i k_i^* u_\ell(\theta_0) \langle E_0^* u, E_0 v \rangle
\]

Result follows.
Fix $v \in \mathbb{R}^n$ and $\alpha \neq v \in \mathbb{R}^n$. Using $uv$ we get a basis $\mathbb{R}^n$.

<table>
<thead>
<tr>
<th>Basis</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E^i$ - standard</td>
<td>$E_i^x u$ $0 \leq i \leq N$</td>
</tr>
<tr>
<td>$E^d$ - dual standard</td>
<td>$\frac{v}{|u|^2} E_i^x u$ $0 \leq i \leq N$</td>
</tr>
<tr>
<td>$E^x$ - standard</td>
<td>$E_i v$ $0 \leq i \leq N$</td>
</tr>
<tr>
<td>$E^x$ - dual standard</td>
<td>$\frac{v}{|u|^2} E_i v$ $0 \leq i \leq N$</td>
</tr>
</tbody>
</table>
Matrices that represent $A$ and $A^*$

<table>
<thead>
<tr>
<th>Basis</th>
<th>$A$</th>
<th>$A^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{F}$ - standard</td>
<td>$B$</td>
<td>$D^*$</td>
</tr>
<tr>
<td>$\mathbb{F}$ - dual standard</td>
<td>$B^\epsilon$</td>
<td>$D^*$</td>
</tr>
<tr>
<td>$\mathbb{F}^*$ - standard</td>
<td>$D$</td>
<td>$B^*$</td>
</tr>
<tr>
<td>$\mathbb{F}^*$ - dual standard</td>
<td>$D$</td>
<td>$B^\epsilon$</td>
</tr>
</tbody>
</table>
Transition matrices

\[
\begin{align*}
\begin{array}{rcl}
\left\{ \frac{\sqrt{\|u\|^2}}{\|u\|} \right\}_{i=0}^N & \rightarrow & K \frac{\|u\|^2}{\sqrt{\|u\|^2}} \rightarrow K^* \frac{\sqrt{\|u\|^2}}{\|u\|} \\
K U \frac{\langle u, v \rangle}{\|v\|^2} & \rightarrow & K U K^* \frac{\langle u, v \rangle}{\sqrt{\|u\|^2}} \\
K^* U^* \frac{\langle u, v \rangle}{\|u\|^2} & \rightarrow & K^* U K \frac{\langle u, v \rangle}{\sqrt{\|u\|^2}} \\
\left\{ \frac{\sqrt{\|v\|^2}}{\|v\|} \right\}_{i=0}^N & \rightarrow & (K^*)^* \frac{\langle u, v \rangle}{\sqrt{\|u\|^2}} \rightarrow \left\{ \frac{\sqrt{\|v\|^2}}{\|v\|} \right\}_{i=0}^N
\end{array}
\end{align*}
\]

Key:

\[
\{ u_i \}_{i=0}^N \rightarrow^M \{ v_i \}_{i=0}^N
\]

means

\[
v_i = \sum_{i=0}^N M_{ij} u_j \quad j = 0, 1, \ldots, N
\]
$K \frac{\langle u, v \rangle}{\|u\|^2}$

$\{ \frac{\langle u, v \rangle}{\|u\|^2}, \frac{E_i^*}{K_i} \}_{i=0}^N$ $\Rightarrow$ $\{ \frac{E_i^*}{K_i} \}_{i=0}^N$

$U \frac{\langle u, v \rangle}{\|u\|^2}$ $\Rightarrow$ $K \frac{\langle u, v \rangle}{\|u\|^2}$

$U^* \frac{\langle u, v \rangle}{\|u\|^2}$ $\Rightarrow$ $K^* \frac{\langle u, v \rangle}{\|u\|^2}$

$U^* K \frac{\langle u, v \rangle}{\|u\|^2}$ $\Rightarrow$ $K^* U K \frac{\langle u, v \rangle}{\|u\|^2}$

$(K^*)^N \frac{\langle u, v \rangle}{\|u\|^2}$ $\Rightarrow$ $K^* \frac{\langle u, v \rangle}{\|u\|^2}$

Key: $\{ u_i \}_{i=0}^N$ $\Rightarrow$ $\{ v_i \}_{i=0}^N$

Means $M_{ij} = \langle u_i, v_j \rangle$ ($0 \leq i, j \leq N$)
Note: If we pick \( \langle \cdot, \cdot \rangle \) such that
\[
\| u \|_2^2 = 1
\]
and pick \( v \) such that
\[
\langle u, v \rangle = 1
\]
then
\[
\| v \|_2^2 = n
\]
by Lemma 260 (iv).

In this case, above diagrams match the ones above Fig 15.8.
Topics

- Given $a_i b_i c_i$
  \[(a_i b_i c_i = 0)\]
  and $PA$

- qRac case
  $Z_{3}-$sym version
  $\rightarrow$ Mock spin model

- $\rho_i = U_i(c_0)$
  $\rho_i / \rho_{i-1} = \text{quad AB}$
  $\Psi, \Delta = \exp(\Psi)$

- Algebras
  $AW(3)$ ($Z_{3}$-sym version)
  $\rightarrow$ (q-Onsager)
  $U_{\text{gal}}$ (left and right embedding)

- DAHA

- Rosenhans
  $\Psi$
  $\Psi$

- Biplabelled case
  $Z_{3}$-sym MV
  Spin model

- qRac case
  $\rightarrow$ Mock spin model

- $\Psi, \Delta = \exp(\Psi)$

- Algebras
  $AW(3)$ ($Z_{3}$-sym version)
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  $U_{\text{gal}}$ (left and right embedding)

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  $\Psi$

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  $Z_{3}$-sym MV
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