Abstract

These are notes on Witten’s 1982 paper \textit{Supersymmetry and Morse Theory}, in which Witten introduced a deformation of the de Rham complex by \( dt = e^{-ht} de^{ht} \) and \( \delta_t = e^{ht} \delta e^{-ht} \), where \( d \) is the usual exterior derivative on forms, \( \delta \) is its adjoint, \( t \in \mathbb{R} \), and \( h \) is a Morse function. He used the deformed complex to obtain relationships between Betti numbers and the Morse indices of the critical points of \( h \).

1 Background in Hodge Theory

Assume throughout that \( M \) is compact, orientable, Riemannian with Levi-Civita connection, and \( n = \dim M \).

Define differential forms as in Bott and Tu.

The \textbf{de Rham cohomology} of \( M \) is

\[
H^k_{\text{dR}}(M) = \frac{\ker d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)}{\im d : \Omega^{k-1}(M) \rightarrow \Omega^k(M)} = \frac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}}.
\]

We call \( \beta_k = \dim H^k_{\text{dR}}(M) \) the \textbf{\( k \)th Betti number} of \( M \).

\textbf{Idea:} \( \Omega^k(M) \) is a priori an infinite dimensional vector space. If we can put an inner product on \( \Omega^k(M) \) and turn it into a Hilbert space, then we can represent each element in the quotient \( H^k_{\text{dR}}(M) \) by an element of smallest norm. Specifically, given a cohomology class \( [\omega] \in H^k_{\text{dR}}(M) \), choose a representative closed form \( \eta \) that is \textit{orthogonal to the subspace of exact } \( k \text{-forms} \) (by which we’ll quotient).
That is, we require that the representative $\eta$ satisfy
\[(\eta, d\psi) = 0\] (1)
for all $\psi \in \Omega^{k-1}(M)$. It turns out that $(\text{im } d)^\perp \cong H^k_{\text{dR}}(M)$.

Continuing to assume that $\Omega^k(M)$ has an inner product, we can define an adjoint $\delta$ to $d$ by
\[(\delta \eta, \psi) = (\eta, d\psi).\]
Now Eq. (1) becomes
\[(\delta \eta, \psi) = 0\] (2)
for all $\psi \in \Omega^{k-1}(M)$. Equivalently,
\[\delta \eta = 0.\]
Hence, we see that the candidate representative closed form $\eta$ must satisfy $d\eta = \delta \eta = 0$.

**Definition 1.** The **Hodge Laplacian** $\Delta : \Omega^k(M) \longrightarrow \Omega^k(M)$ is given by
\[
\Delta = \pm (d\delta + \delta d).
\]
The vector space of **harmonic forms** is
\[
\mathcal{H}^k(M) := \ker \Delta.
\]

**1 Proposition:**
1. \[\Delta \alpha = 0 \quad \iff \quad d\alpha = \delta \alpha = 0\]
2. $\Delta$ is self-adjoint, i.e., for all $\alpha, \beta \in \Omega^k(M)$,
\[\langle \Delta \alpha, \beta \rangle = \langle \alpha, \Delta \beta \rangle.\]
3. $\Delta$ is non-negative. \(\diamondsuit\)

**Proof**
\[
\langle \Delta \alpha, \alpha \rangle = \langle (\delta d + d\delta) \alpha, \alpha \rangle = \langle d\alpha, d\alpha \rangle + \langle \delta \alpha, \delta \alpha \rangle
\]

Now we see that the candidate representative closed form $\eta$ must be harmonic, i.e., $\eta \in \mathcal{H}^k(M)$.

Since all harmonic forms are closed, we have a natural map:
\[
\mathcal{H}^k(M) \longrightarrow H^k_{\text{dR}}(M).
\]
1 Theorem (Hodge, 1935): 
This map is an isomorphism. Hence, every cohomology class $[\omega]$ has a unique harmonic representative $\eta \in \mathcal{H}^k(M)$. Moreover, we have the decomposition
\[
\Omega^k(M) = d(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M)) \oplus \mathcal{H}^k(M).
\]

Some comments:
1. Since $\Omega^k(M)$ contains only smooth forms and not all $L^2$ $k$-forms, it is not a Hilbert space. One must consider Sobolev spaces and talk about the decomposition for Fredholm operators.
2. One achieves the desired inner product by defining the Hodge star operator
\[
*: \Omega^k(M) \rightarrow \Omega^{n-k}(M)
\]
on forms. $*$ is an isomorphism that satisfies
\[
*^2 = (-1)^{k(n-k)}
\]
and for $\alpha, \beta \in \Omega^k(M)$, we define
\[
(\alpha, \beta) := \int_M \alpha \wedge *\beta.
\]
Then
\[
\delta = \pm *d*.
\]

2 Morse Theory

- A map $f : M \rightarrow \mathbb{R}$ has a critical point at $p \in M$ if $df(p) = 0$.
- The Hessian of $f$ at $p$ is the symmetric bilinear functional
\[
\text{Hess}_p f : T_pM \times T_pM \rightarrow \mathbb{R}
\]
defined by
\[
\text{Hess}_p(X, Y) := \tilde{X}_p(\tilde{Y}(f)),
\]
where $\tilde{X}$ and $\tilde{Y}$ are any extension of $X$ and $Y$ to $\mathfrak{X}(M)$.
- We say the $p$ is non-degenerate if $\dim \ker \text{Hess}_p = 0$.
- The index of $f$ at $p$ is
\[
\lambda_p = \dim \{ X \in T_pM : \text{Hess}_p(X, Y) < 0 \text{ for all } Y \in T_pM \}.
\]
2 Theorem (Morse’s Lemma):

If \( p \) is a non-degenerate critical point of \( f : M \to \mathbb{R} \) with index \( \lambda_p \), there is a neighborhood \( U \) of \( p \) and a coordinate system \( x : U \to \mathbb{R}^n \) such that, for \( q \in U \),

\[
f(q) = f(p) - \sum_{i=1}^{\lambda_p} (x^i(q))^2 + \sum_{i=\lambda_p+1}^{n} (x^i(q))^2.
\]

Definition 2. A **Morse function** is a smooth function \( f : M \to \mathbb{R} \) such that all the critical points are interior and non-degenerate.

Some comments:

1. By Morse’s Lemma, all critical points of a Morse function are isolated.

2. An involved argument using Sard’s Theorem provides for the existence (and abundance) of Morse functions.

3. The main idea here is that we can use Morse functions to determine the curvature behavior at critical points, by essentially the “Second Derivative Test,” i.e., approximation by quadratic forms. By putting a gradient flow on \( M \), we can “cancel” extraneous critical points in pairs until we are left only with those that signify some change in the topology of \( M \) as we move through the level sets of \( f \).

3 Theorem (Weak Form of the Morse Inequalities):

Let \( M_\lambda \) denote the number of critical points of index \( \lambda \). Then

\[
M_\lambda \geq \beta_\lambda.
\]
3 Witten Deformation

**Definition 3.** Let $h : M \to \mathbb{R}$ be a smooth Morse function and $t \in \mathbb{R}$. Define

\[ d_t := e^{-ht} de^{ht} \quad \delta_t := e^{ht} de^{-ht} \quad \Delta_t := d_t \delta_t + \delta_t d_t \]

and

\[ \mathcal{H}^k_t(M) := \ker \Delta_t. \]

We immediately have

\[ d_t^2 = \delta_t^2 = 0. \]

Define

\[ H^k_t(M) = \frac{\ker d_t : \Omega^k(M) \to \Omega^{k+1}(M)}{\text{im } d_t : \Omega^{k-1}(M) \to \Omega^k(M)} \]

and $\beta_k(t) = \dim H^k_t$. It is a fact that

\[ \beta_k(t) = \beta_k \] for all $t \in \mathbb{R}$.

It follows that

\[ \beta_k = \dim \mathcal{H}^k_t(M). \]

We’ll use the nice fact that:

*The spectrum of $\Delta_t$ simplifies dramatically for large $t$.*

Let’s see why the critical points of $h$ enter:

Choose a local orthonormal frame $\{a_k\}_{k=1}^n$. We can consider the action

\[ a^k : \Omega^k(M) \to \Omega^{k-1}(M) \]

\[ a^k : \psi \mapsto a^k \cdot \psi = i_{a_k}(\psi). \]

Denote the adjoint operator by

\[ a^{k*} : \Omega^{n-k}(M) \to \Omega^{n-k+1}(M) \]

\[ a^{k*} : \psi \mapsto a^{k*} \cdot \psi = (a^k)^\flat \wedge \psi. \]

On a Riemannian manifold, we can talk about the second covariant derivative of $h$ with components $\frac{D^2 h}{D\phi^i D\phi^j}$ in the basis dual to the $a^k$. 

4 Theorem:
With the notation as above, we have
\[ \Delta_t = \Delta + t^2(dh)^2 + \sum_{i,j} \frac{D^2 h}{D\phi^i D\phi^j} [a^i, a^j], \]
where
\[ (dh)^2 = g^{ij} \frac{\partial h}{\partial \phi^i} \frac{\partial h}{\partial \phi^j}. \]

So for large \( t \), \( \Delta_t \) is only small near critical points of \( h \), where \( dh = 0 \), and, hence, the forms in \( \mathcal{H}_t^k(M) \) are concentrated near the critical points of \( h \). Near a critical point \( p^a \), \( \Delta_t \) can be approximated as
\[ \Delta_t \approx \sum_i \left( -\frac{d^2}{d\phi^2_i} + t^2 \lambda_i^2 \phi_i^2 + t \lambda_i [a^i, a^i] \right). \]

The first two terms are essentially the Hamiltonian of the harmonic oscillator in quantum mechanics. (The last term is like the central term in \( sl(2n).(?)) \) It’s suggestive to rewrite this, therefore, as
\[ \Delta_t \approx \sum_i (H_i + t \lambda_i K_i) \]
where
\[ H_i = -\frac{d^2}{d\phi^2_i} + t^2 \lambda_i^2 \phi_i^2 \quad K_i = [a^i, a^i]. \]

Then \( H_i \) and \( K_i \) can be simultaneously diagonalized. By standard QM calculations, one finds that the eigenvalues of \( \Delta_t \) are
\[ t \sum_i (|\lambda_i| (1 + 2N_i) + \lambda_i n_i) \quad N_i = 0, 1, 2, \ldots \quad n_i = \pm 1. \]

For a zero eigenvalue, therefore, we need \( N_i = 0 \) for all \( i \), and \( n_i = +1 \) iff \( \lambda_i < 0 \). So

Expanding around any given critical point, \( \Delta_t \) has precisely one zero eigenvalue, corresponding to a \( k \)-form if the critical point has index \( k \).

4 Supersymmetry Algebras