Euler’s Method and Applications I

1. (a) What’s the main idea of the Euler’s Method?

(b) If we want to use Euler’s Method to approximate a solution to a differential equation, what should we have to start?
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2. For the differential equation $\frac{dy}{dx} = x^2 + \sin(x)y^2$, and a given initial value condition $y(0) = 1$.
   (a) Can you solve this equation?

   (b) Let $f(x, y) = \frac{dy}{dx} = x^2 + \sin(x)y^2$, $x_0 = 0$, $y_0 = y(x_0) = 1$. Compute $f(x_0, y_0)$. What is the meaning of this number?

   (c) Set the step size $h = \frac{\pi}{2}$. Use the Euler’s Method to approximate $y(\pi)$.

   (d) Use the Euler’s Method to approximate $y(\pi)$ for the step size $h = \frac{\pi}{4}$.
(e) Compare the two approximations for \( y(\pi) \) above. Which one do you think is a better approximation? Why?

3. Let \( P(t) \) denote the population of algae in Lake Mendota, where \( t \) is measured in years. Say we want to find a model for the population. Researchers have found that since the year 2000, the change in population from year to year is proportional to the current population multiplied by the number of years since 2000 (with the proportionality constant called \( k \)). Given that there were \( 10^4 \) algae in the lake in 2000 and that the population is known to double every 7 years, find an explicit equation for \( P(t) \) in the following way:

(a) Set up a differential equation describing \( P \). (Hint: This will involve \( k \))

(b) Find the general solution to this differential equation. (Hint: This will now involve two constants, one of which is \( k \))

(c) Use the initial value to solve for the variable other than \( k \).

(d) Given the initial value and the fact that the population doubles every 7 years, find \( k \).

Solutions:

1. (a) Approximate the solutions by a series of short segments of straight lines.
   (b) Equation: \( \frac{dy}{dx} = f(x, y) \)
       - Initial point: \( x_0, y_0 \)
       - Final point: \( x_n \) (\( n \) is the number of steps)
       - Step size: \( h \)
2. (a) No.
   
   (b) \( f(x_0, y_0) = 0 \), the slope at the point \((0, 0)\).

   (c) 2 steps:
   - \( x_0 = 0, \ y_0 = 1, \ f(x_0, y_0) = 0, \ y_1 = 1 \)
   - \( x_1 = \pi/2, \ y_1 = 1, \ f(x_1, y_1) = \pi^2/4 + 1, \ y_2 = \pi^3/8 + \pi/2 + 1 \)
   
   So \( y(\pi) \approx y_2 = \pi^3/8 + \pi/2 + 1 \).

   (d) \(-17.56\)

   (e) We cannot compare these two approximations in this case, since we have no idea about the real solutions.

3. (a) \( \frac{dP}{dt} = kP \)

   (b) \( P(t) = Ce^{kt} \)

   (c) \( C = 4 \ln 10 \)

   (d) \( k = \frac{\ln 2}{7} \)